OPENING NODES IN THE DPW METHOD: CO-PLANAR CASE

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Abstract: we combine the DPW method and opening nodes to construct embedded surfaces of positive constant mean curvature with Delaunay ends in euclidean space, with no limitation to the genus or number of ends.

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1. Introduction

In [5], Dorfmeister, Pedit and Wu have shown that harmonic maps from a Riemann surface to a symmetric space admit a Weierstrass-type representation, which means that they can be represented in terms of holomorphic data. In particular, surfaces with constant mean curvature one (CMC-1 for short) in euclidean space admit such a representation, owing to the fact that the Gauss map of a CMC-1 surface is a harmonic map to the 2-sphere. This representation is now called the DPW method and has been widely used to construct CMC-1 surfaces in $\mathbb{R}^3$ and also constant mean curvature surfaces in homogeneous spaces such as the sphere $\mathbb{S}^3$ or hyperbolic space $\mathbb{H}^3$: see for example [4, 6, 9, 10, 11, 14, 15, 22].

The input data for the DPW method is called the DPW potential. In principle, all CMC surfaces can be obtained by the DPW method. But in practice, one has to solve a Monodromy Problem, akin to the Period Problem for the construction of minimal surfaces via the Weierstrass Representation. So in general the topology of the constructed examples is limited or symmetries are imposed in order to reduce the number of equations to be solved. In contrast, Kapouleas [13] has constructed embedded CMC surfaces with no limitation on the genus or number of ends by gluing round spheres and pieces of Delaunay surfaces, using PDE methods. It seems an interesting question to see whether such gluing constructions can be achieved by the DPW method.

In [28], we proposed a DPW potential for CMC n-noids: genus zero CMC-1 surfaces with n Delaunay-type ends. They look like a round sphere with n half-Delaunay surfaces with small necksize attached at prescribed points. They are a particular case of the construction of Kapouleas in [13]. The potential is natural, in the sense that it is a perturbation of the standard spherical potential. This potential has been adapted to minimal surfaces in $\mathbb{H}^3$ and AdS^3 in [1] and CMC>1 surfaces in $\mathbb{H}^3$ in [21].

In [29], we proposed a DPW potential for another type of CMC n-noids which look like a minimal n-noid (a genus zero minimal surface with n catenoidal ends) whose catenoidal ends have been replaced by Delaunay ends. They had already been constructed by Mazzeo and Pacard in [18] using PDE methods. The potential is derived in a natural way from the Weierstrass data of the minimal n-noid. It has also been adapted to CMC>1 surfaces in $\mathbb{H}^3$ in [21].

Our goal in this paper is to propose a DPW potential for the higher genus surfaces constructed by Kapouleas in [13] in the case where all the centers of the spheres to be glued together are in the same plane. The resulting CMC surfaces are invariant by symmetry with respect to that plane. The symmetry allows us to take advantage of the fact that the standard holomorphic frame for Delaunay surfaces is unitary on the unit circle, which is a big asset for the resolution of the Monodromy Problem.

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The underlying Riemann surface is defined by opening nodes, which is a standard model for Riemann surfaces with “small necks”. The theory of opening nodes has been used by the author to construct minimal surfaces in euclidean space via the classical Weierstrass Representation (see for example [24] or [25]) or CMC-1 surfaces in hyperbolic space via Bryant Representation [27].

One difficulty with the DPW method is that unlike the Weierstrass data of minimal surfaces, the DPW potential has little geometric content so it is hard to guess a candidate for the construction of CMC surfaces with given geometric features. The heuristic that we follow is that the DPW potential should be a perturbation of the spherical potential where the surface is close to a round sphere and of the catenoidal potential where the surface has small catenoidal necks.

This paper opens up the possibility of opening nodes in the DPW method. We hope the ideas developed in this paper will be useful to the contruction of minimal and CMC surfaces in other space forms.

Remark 1. In an unpublished paper [30], I proposed a DPW potential for all the surfaces constructed by Kapouleas in [13], with no symmetry assumption. The potential was, however, quite complicated and hardly natural, and the paper was long and technical. [30] will not be published in its present form, as I hope a simpler potential will be found in the general case. The result of Appendix B of [30] has been moved to the appendix of the present paper to make it self-contained. For the interested reader, the result of Appendix A of [30] has been moved to [12] where it is needed.

2. Main result

Our goal is to contruct CMC surfaces by gluing spheres and half-Delaunay surfaces. The layout of these pieces is encoded by a weighted graph in the horizontal plane.

Definition 1. A horizontal weighted graph Γ is the following data:

- A finite number of points $v_j \in \mathbb{R}^2$ for $j \in J$, called vertices. Here $J \subset \mathbb{N}^*$ is a finite set used to index vertices.
- A symmetric subset $E \subset (J \times J) \setminus \Delta$ where Δ is the diagonal of $J \times J$, whose elements are called edges. Two vertices v_j and v_k are adjacent if $(j,k) \in E$.
- A finite set of half-lines $\Delta_{jk} \subset \mathbb{R}^2$ for $(j,k) \in R$, called rays, such that Δ_{jk} has endpoint v_j. Here $R \subset J \times (\mathbb{N}^* \setminus J)$ is a finite set used to index rays.
- Each edge or ray is given a non-zero weight τ_{jk}, $(j,k) \in E \cup R$, with $\tau_{jk} = \tau_{kj}$ for $(j,k) \in E$.

For $j \in J$, we denote $E_j = \{k \in J : (j,k) \in E\}$ the set of edges issued from the vertex v_j, and $R_j = \{k \in \mathbb{N}^*, (j,k) \in R\}$ the indices of the rays issued from the vertex v_j. Also we denote $E^+ = \{(j,k) \in E : j < k\}$.

Given a horizontal weighted graph Γ with length-2 edges, we can construct a singular CMC-1 surface M_0 as follows. We identify $\mathbb{R}^2$ with the horizontal plane $x_3 = 0$.

- For $j \in J$, place a radius-1 sphere centered at the vertex v_j, so if v_j and v_k are adjacent, the corresponding spheres are tangent.
- For each $(j,k) \in R$, place an infinite chain of radius-1 spheres with centers on Δ_{jk} at even distance from v_j.

Our goal in this paper is to construct a family of CMC-1 surfaces $(M_t)_{0<t<\epsilon}$ by desingularizing M_0, replacing all tangency points between adjacent spheres by catenoidal necks of size $\simeq t\tau_{jk}$ (see Figure 1). This is only a heuristic way to describe the result, and is not the way we will construct M_t (although this is how Kapouleas does in [13]).

For the construction to succeed, the weighted graph Γ must satisfy a balancing condition. For $(j,k) \in E$, we denote $\ell_{jk} = |v_j - v_k|$ and u_{jk} the unitary vector $(v_k - v_j)/\ell_{jk}$, so $u_{kj} = -u_{jk}$. For $(j,k) \in R$, we denote u_{jk} the unitary vector in the direction of the ray Δ_{jk}.

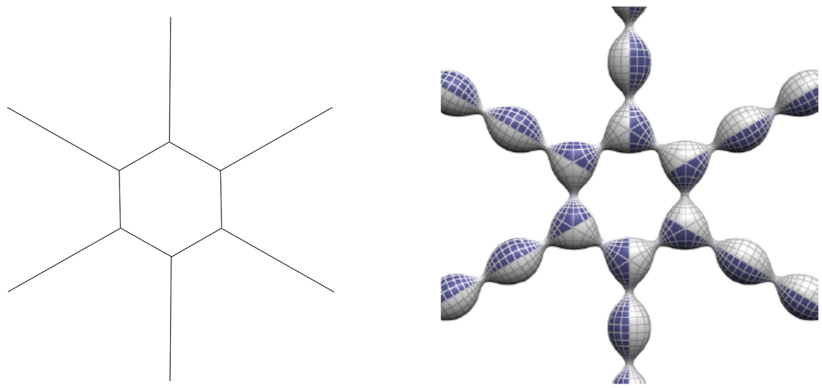

FIGURE 1. Left: a balanced graph with 6 edges and 6 rays. All edges and rays have weight 1. Right: a CMC-1 surface of genus 1 with 6 Delaunay-type ends in the corresponding family. Computer image by N. Schmitt [2].

Definition 2. For $j \in J$, we define the force F_i on the vertex v_j by

$$F_j = \sum_{k \in E_j \cup R_j} \tau_{jk} u_{jk}. \tag{1}$$

A horizontal weighted graph Γ is balanced if $F_j = 0$ for all $j \in J$.

To solve our problem, we need to perturb Γ in order to prescribe small variations of edge-lengths and forces. The parameters available to deform Γ are the vertices $v_j \in \mathbb{R}^2$ for $j \in J$, the unitary vectors u_{jk} for $(j,k) \in R$ and the weights $\tau_{jk} \in \mathbb{R}$ for $(j,k) \in E^+ \cup R$.

Definition 3. A horizontal weighted graph Γ is non-degenerate if the jacobian of the map $\big((F_j)_{j\in J}, (\ell_{jk})_{(j,k)\in E^+}\big)$ with respect to the above parameters is onto.

Theorem 1. *Let Γ be a balanced, non-degenerate horizontal weighted graph with length-2 edges. There exists a smooth 1-parameter family of immersed CMC-1 surfaces $(M_t)_{0<t<\epsilon}$ with the following properties:*

1. *(M_t) converges to M_0 as $t \to 0$. The convergence is for the Hausdorf distance on compact sets of $\mathbb{R}^3$.*
2. *M_t is homeomorphic to the boundary of a small tubular neighborhood of Γ.*
3. *M_t is symmetric with respect to the horizontal plane.*
4. *For each $(j,k) \in R$, M_t has a Delaunay end with weight $\simeq 2\pi t \tau_{jk}$ and whose axis converges as $t \to 0$ to the ray Δ_{jk}.*
5. *If all weights are positive, then M_t is Alexandrov-embedded.*
6. *If moreover Γ is pre-embedded, then M_t is embedded.*

Definition 4. Following Kapouleas (Definition 2.2 in [13]), we say that Γ is pre-embedded if the distance between any two edges or rays which have no common endpoint is greater than 2 and the angle between any two edges or rays with a common endpoint is greater than $60°$.

Remark 2. A balanced graph with even length edges can be transformed into a graph with length-2 edges by adding vertices, transforming an edge of length $2k$ into k edges of length 2, with the same weight. Clearly the resulting graph is balanced, and it is easy to see that non-degeneracy is preserved.

3. Background

3.1. **Functional spaces.** The DPW method uses loop groups, which are groups of smooth functions from the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ to a matrix group. The circle variable is denoted λ. The DPW method is usually formulated in the category of smooth maps, but since we plan to use the Implicit Function Theorem, we need a Banach space. We adopt the following choice, following [28, 29].

Fix some $\rho > 1$ and let $\mathbb{D}_\rho \subset \mathbb{C}$ be the disk $|\lambda| < \rho$ and $\mathbb{A}_\rho \subset \mathbb{C}$ the annulus $\rho^{-1} < |\lambda| < \rho$. We decompose a smooth function $f : \mathbb{S}^1 \to \mathbb{C}$ in Fourier series

$$f(\lambda) = \sum_{i\in\mathbb{Z}} f_i \lambda^i$$

and define

$$\|f\| = \sum_{i\in\mathbb{Z}} |f_i| \rho^{|i|}$$

Let $\mathcal{W}$ be the space of functions f with finite norm. This is a Banach algebra, owing to the fact that the weight $\rho^{|i|}$ is submultiplicative (see Section 4 in [8]). Functions in $\mathcal{W}$ extend holomorphically to $\mathbb{A}_\rho$.

We define $\mathcal{W}^{\geq 0}$, $\mathcal{W}^{>0}$, $\mathcal{W}^{\leq 0}$ and $\mathcal{W}^{<0}$ as the subspaces of functions f such that $f_i = 0$ for $i < 0$, $i \leq 0$, $i > 0$ and $i \geq 0$, respectively. Functions in $\mathcal{W}^{\geq 0}$ extend holomorphically to the disk $\mathbb{D}_\rho$ and satisfy $|f(\lambda)| \leq \|f\|$ for all $\lambda \in \mathbb{D}_\rho$. We write $\mathcal{W}^0 \sim \mathbb{C}$ for the subspace of constant functions, so we have a direct sum $\mathcal{W} = \mathcal{W}^{<0} \oplus \mathcal{W}^0 \oplus \mathcal{W}^{>0}$. A function f will be decomposed as $f = f^- + f^0 + f^+$ with $(f^-, f^0, f^+) \in \mathcal{W}^{<0} \times \mathcal{W}^0 \times \mathcal{W}^{>0}$ (and of course $f^0 = f_0$).

We define the conjugation operator by

$$\overline{f}(\lambda) = \overline{f(\overline{\lambda})} = \sum_{i\in\mathbb{Z}} \overline{f_i} \lambda^i.$$

We denote $\mathrm{Re}(f) = \frac{1}{2}(f + \overline{f})$ and $\mathrm{Im}(f) = \frac{1}{2i}(f - \overline{f})$ and define $\mathcal{W}_\mathbb{R}$ as the subspace of functions in $\mathcal{W}$ such that $\mathrm{Im}(f) = 0$, and $\mathcal{W}_\mathbb{R}^{\geq 0} = \mathcal{W}_\mathbb{R} \cap \mathcal{W}^{\geq 0}$.

We also define the star operator by

$$f^*(\lambda) = \overline{f(1/\overline{\lambda})} = \sum_{i\in\mathbb{Z}} \overline{f_{-i}} \lambda^i.$$

The involution $f \mapsto f^*$ exchanges $\mathcal{W}^{\geq 0}$ and $\mathcal{W}^{\leq 0}$. We have $\lambda^* = \lambda^{-1}$ and $c^* = \overline{c}$ if c is a constant. A function f is real on the unit circle if and only if $f = f^*$. Note that conjugation and star commute.

There is a theory of holomorphic functions between complex Banach space, which retain most properties of holomorphic functions of several variables. A good reference is [3].

3.2. **Loop groups.**

- If G is a matrix Lie group, we denote ΛG the Banach Lie group of maps $\Phi : \mathbb{S}^1 \to G$ whose entries are in $\mathcal{W}$.
- If $\mathfrak{g}$ is the Lie algebra of G, the Lie algebra of ΛG is the set of maps $\varphi : \mathbb{S}^1 \to \mathfrak{g}$ whose entries are in $\mathcal{W}$ and is denoted $\Lambda\mathfrak{g}$.
- $\Lambda_+ SL(2,\mathbb{C}) \subset \Lambda SL(2,\mathbb{C})$ is the subgroup of maps B whose entries are in $\mathcal{W}^{\geq 0}$, with $B\mid_{\lambda=0}$ upper triangular.
- $\Lambda_+^\mathbb{R} SL(2,\mathbb{C}) \subset \Lambda_+ SL(2,\mathbb{C})$ is the subgroup of maps B such that $B\mid_{\lambda=0}$ has positive entries on the diagonal.

The following result is the corner stone of the DPW method. It is usually formulated for smooth loops [19], but adapts with no difficulty to loops with entries in $\mathcal{W}$ (see details in Section 3.6 of [29]).

Theorem 2 (Iwasawa decomposition)**.** *The multiplication $\Lambda SU(2) \times \Lambda_+^\mathbb{R} SL(2,\mathbb{C}) \to \Lambda SL(2,\mathbb{C})$ is a smooth diffeomorphism (in the sense of smooth maps between Banach manifolds). The unique splitting* *of an element* $\Phi \in \Lambda SL(2,\mathbb{C})$ *as* $\Phi = FB$ *with* $F \in \Lambda SU(2)$ *and* $B \in \Lambda_+^{\mathbb{R}} SL(2,\mathbb{C})$ *is called Iwasawa decomposition.* F *is called the unitary factor of* Φ *and denoted* $\mathrm{Uni}(\Phi)$*.* B *is called the positive factor and denoted* $\mathrm{Pos}(\Phi)$*.*

3.3. **The DPW method.** In the DPW method, one identifies $\mathbb{R}^3$ with the Lie algebra $\mathfrak{su}(2)$ by

$$(x_1, x_2, x_3) \in \mathbb{R}^3 \quad \sim \quad -i \begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \in \mathfrak{su}(2).$$

The input data for the DPW method is a quadruple $(\Sigma, \xi, z_0, \phi_0)$ where Σ is a Riemann surface, ξ is a $\Lambda\mathfrak{sl}(2,\mathbb{C})$-valued holomorphic 1-form on Σ of the following special form

$$\xi = \begin{pmatrix} \alpha & \lambda^{-1}\beta \\ \gamma & -\alpha \end{pmatrix} \tag{2}$$

where α, β, γ are $\mathcal{W}^{\geq 0}$-valued holomorphic 1-forms on Σ, $z_0 \in \Sigma$ is a base point and $\phi_0 \in \Lambda SL(2,\mathbb{C})$ is an initial condition. ξ is called the DPW potential. If Σ is simply connected, the DPW method is the following procedure:

- Solve the Cauchy Problem on Σ:

$$\begin{cases} d_z\Phi = \Phi\xi \\ \Phi(z_0) = \phi_0 \end{cases} \tag{3}$$

  to obtain a solution $\Phi : \Sigma \to \Lambda SL(2,\mathbb{C})$.
- Compute the Iwasawa decomposition $(F(z), B(z))$ of $\Phi(z)$ for $z \in \Sigma$.
- Define $f : \Sigma \to \mathfrak{su}(2) \sim \mathbb{R}^3$ by the Sym-Bobenko formula:

$$f(z) = \mathrm{Sym}(F(z)) = -2i\,\frac{\partial F(z)}{\partial\lambda} F(z)^{-1}\mid_{\lambda=1}. \tag{4}$$

  Then f is a CMC-1 (branched) conformal immersion. f is regular at z (meaning unbranched) if and only if $\beta^0(z) \neq 0$. Its Gauss map is given by

$$N(z) = \mathrm{Nor}(F(z)) = -i\,F(z) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} F(z)^{-1}\mid_{\lambda=1}. \tag{5}$$

  The DPW method actually constructs a moving frame for f and the differential of f is given by

$$df(z) = 2i\,B_{11}^0(z)^2 F(z) \begin{pmatrix} 0 & \beta^0(z) \\ \overline{\beta^0(z)} & 0 \end{pmatrix} F(z)^{-1}\mid_{\lambda=1}. \tag{6}$$

3.4. **The Monodromy Problem.** If Σ is not simply connected, lift the DPW potential ξ to the universal cover $\widetilde{\Sigma}$ of Σ and choose a point $\widetilde{z_0}$ in the fiber of z_0. Solve the Cauchy Problem $d\Phi = \Phi\xi$ in $\widetilde{\Sigma}$ with initial condition $\Phi(\widetilde{z_0}) = \phi_0$ to define $\Phi : \widetilde{\Sigma} \to \Lambda SL(2,\mathbb{C})$. The DPW method produces an immersion $f : \widetilde{\Sigma} \to \mathbb{R}^3$.

For $\gamma \in \pi_1(\Sigma, z_0)$, let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{\Sigma}$ such that $\widetilde{\gamma}(0) = \widetilde{z_0}$. The monodromy of Φ with respect to γ is

$$\mathcal{M}(\Phi, \gamma) = \Phi(\widetilde{\gamma}(1))\Phi(\widetilde{\gamma}(0))^{-1}$$

The standard condition which ensures that the immersion f descends to a well defined immersion on Σ is the following system of equations, called the Monodromy Problem:

$$\forall\gamma \in \pi_1(\Sigma, z_0) \quad \begin{cases} \mathcal{M}(\Phi,\gamma) \in \Lambda SU(2) & (i) \\ \mathcal{M}(\Phi,\gamma)\mid_{\lambda=1} = \pm I_2 & (ii) \\ \frac{\partial}{\partial\lambda}\mathcal{M}(\Phi,\gamma)\mid_{\lambda=1} = 0 & (iii) \end{cases} \tag{7}$$

We will formulate the Monodromy Problem using the notion of principal solution (see Chapter 3.4 in [23]).

Definition 5. Let $\gamma : [0,1] \to \Sigma$ be a path, not necessarily closed. Let $Y : [0,1] \to \Lambda SL(2,\mathbb{C})$ be the solution of the Cauchy Problem

$$\begin{cases} Y'(s) = Y(s)\,\xi(\gamma(s))(\gamma'(s)) \\ Y(0) = I_2 \end{cases}$$

The principal solution of ξ with respect to γ is $\mathcal{P}(\xi,\gamma) = Y(1)$.

In other words, $\mathcal{P}(\xi,\gamma)$ is the value at $\gamma(1)$ of the analytical continuation along γ of the solution of the Cauchy Problem (3) with initial condition $\Phi(\gamma(0)) = I_2$. If p, q are two points on Σ and the path γ from p to q is clear from the context, we will sometime write $\mathcal{P}(\xi,p,q)$ for $\mathcal{P}(\xi,\gamma)$. The principal solution has the following properties, which follow easily from its definition:

- $\mathcal{P}(\xi,\gamma)$ only depends on the homotopy class of γ.
- The principal solution is a morphism for the product of paths: If γ_1 and γ_2 are two paths such that $\gamma_1(1) = \gamma_2(0)$ then
$$\mathcal{P}(\xi,\gamma_1\gamma_2) = \mathcal{P}(\xi,\gamma_1)\mathcal{P}(\xi,\gamma_2).$$
- If $\psi : \Sigma_1 \to \Sigma_2$ is a holomorphic map, ξ is a potential on Σ_2 and γ is a path on Σ_1, then
$$\mathcal{P}(\psi^*\xi,\gamma) = \mathcal{P}(\xi,\psi(\gamma)).$$
- If $\sigma : \Sigma_1 \to \Sigma_2$ is a anti-holomorphic map, then
$$\mathcal{P}(\overline{\sigma^*\xi},\gamma) = \overline{\mathcal{P}(\xi,\sigma(\gamma))}.$$

Back to the DPW method, if the initial condition is $\Phi(z_0) = I_2$, which will be the case in this paper, the Monodromy Problem is equivalent to the following problem:

$$\forall \gamma \in \pi_1(\Sigma, z_0) \quad \begin{cases} \mathcal{P}(\xi,\gamma) \in \Lambda SU(2) & (i) \\ \mathcal{P}(\xi,\gamma)\mid_{\lambda=1} = \pm I_2 & (ii) \\ \frac{\partial}{\partial\lambda}\mathcal{P}(\xi,\gamma)\mid_{\lambda=1} = 0 & (iii) \end{cases} \tag{8}$$

3.5. **Gauging and the Regularity Problem.**

Definition 6. A gauge on Σ is a holomorphic map $G : \Sigma \to \Lambda_+ SL(2,\mathbb{C})$.

Let Φ be a solution of $d\Phi = \Phi\xi$ and G be a gauge. Let $\widehat{\Phi} = \Phi G$. Then Φ and $\widehat{\Phi}$ define the same immersion f via the DPW method. The gauged potential is

$$\widehat{\xi} := \widehat{\Phi}^{-1} d\widehat{\Phi} = G^{-1}\xi G + G^{-1} dG$$

and is denoted $\xi \cdot G$, the dot denoting the action of the gauge group on the potential. Gauging does not change the monodromy of Φ.

Definition 7. We say that ξ is regular at $p \in \Sigma$ if $\beta^0(p) \neq 0$. This ensures that the immersion f is unbranched at p.

In general Σ is a compact Riemann surface $\overline{\Sigma}$ minus a finite number of points, and the potential ξ extends meromorphically to $\overline{\Sigma}$.

Definition 8. We say that a pole p of ξ is an apparent singularity if there exists a meromorphic gauge G, defined in a neighborhood of p, such that $\xi \cdot G$ extends holomorphically at p and is regular. This ensures that the immersion f extends analytically at p.

Our potential will have two kinds of poles: some of them will be ends of the immersion f, the others will be apparent singularities. Note that ξ must have apparent singularities at the zeros of β^0 for f to be regular. If $\overline{\Sigma}$ has positive genus, β^0 must have zeros on $\overline{\Sigma}$ so apparent singularities cannot be avoided.

3.6. **Dressing and rigid motions.** Let Φ be a solution of the Cauchy Problem (3). Let $H \in \Lambda SU(2)$ and define $\widetilde{\Phi}(z) = H\Phi(z)$. Then $\widetilde{\Phi}$ solves $d\widetilde{\Phi} = \widetilde{\Phi}\xi$ and the Iwasawa decomposition of $\widetilde{\Phi}$ is $\widetilde{F} = HF$ and $\widetilde{B} = B$. The Sym-Bobenko formula gives

$$\widetilde{f}(z) = \mathrm{Sym}(\widetilde{F}(z)) = \left(HfH^{-1} - 2i\frac{\partial H}{\partial \lambda}H^{-1} \right)|_{\lambda=1}\ .$$

Consequently, we define a left action of $\Lambda SU(2)$ on $\mathfrak{su}(2)$ by

$$H \cdot x = \left(HxH^{-1} - 2i\frac{\partial H}{\partial \lambda}H^{-1} \right)|_{\lambda=1}\ . \tag{9}$$

The action is by rigid motion and $\widetilde{f} = H \cdot f$. The Monodromy Problems for Φ and $H\Phi$ are equivalent because $H \in \Lambda SU(2)$.

3.7. **Spherical and catenoidal potentials.** Delaunay surfaces are obtained from the following standard potential on $\mathbb{C}^*$:

$$\xi = \begin{pmatrix} 0 & \lambda^{-1}r + s \\ \lambda r + s & 0 \end{pmatrix} \frac{dz}{z}$$

with initial condition $\Phi(1) = I_2$, where r, s are non-zero real numbers such that $r + s = \frac{1}{2}$. There are two limiting cases of interest to us:

- Spherical limit: $(r, s) = (1/2, 0)$ gives

  $$\xi^S = \begin{pmatrix} 0 & \lambda^{-1}/2 \\ \lambda/2 & 0 \end{pmatrix} \frac{dz}{z}$$

  which we call the spherical Delaunay potential. The corresponding solution is

  $$\Phi^S(z) = \frac{1}{2\sqrt{z}} \begin{pmatrix} z+1 & \lambda^{-1}(z-1) \\ \lambda(z-1) & z+1 \end{pmatrix}.$$

  It Iwasawa decomposition is

  $$F^S(z) = \frac{1}{\sqrt{2}\sqrt{1+|z|^2}} \begin{pmatrix} \overline{z}+1 & \lambda^{-1}(z-1) \\ \lambda(1-\overline{z}) & z+1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

  $$B^S(z) = \frac{1}{\sqrt{2|z|}\sqrt{1+|z|^2}} \begin{pmatrix} 2|z| & 0 \\ \lambda(|z|^2-1) & 1+|z|^2 \end{pmatrix}$$

  where $\theta = \arg(z)$. The Sym-Bobenko formula (4) and Equation (5) give

  $$f^S(z) = \frac{-i}{1+|z|^2} \begin{pmatrix} -|z-1|^2 & 1-|z|^2-z+\overline{z} \\ 1-|z|^2+z-\overline{z} & |z-1|^2 \end{pmatrix} \sim \frac{1}{1+|z|^2}\left(1-|z|^2, -2\,\mathrm{Im}(z), |z-1|^2\right)$$

  $$N^S(z) = \frac{-i}{1+|z|^2} \begin{pmatrix} -z-\overline{z} & |z|^2-1+z-\overline{z} \\ |z|^2-1+\overline{z}-z & z+\overline{z} \end{pmatrix} \sim \frac{1}{1+|z|^2}\left(|z|^2-1, 2\,\mathrm{Im}(z), 2\,\mathrm{Re}(z)\right).$$

  Consider the rigid motion

  $$\Psi(x_1, x_2, x_3) = (1-x_3, -x_2, -x_1). \tag{10}$$

  Then

  $$\Psi \circ f^S(z) = \frac{1}{1+|z|^2}\left(2\,\mathrm{Re}(z), 2\,\mathrm{Im}(z), |z|^2-1\right) = \pi^{-1}(z) \tag{11}$$

where $\pi : \mathbb{C}\cup\{\infty\} \to \mathbb{S}^2$ is the stereographic projection from the north pole. The poles at 0 and ∞ are of course apparent singularities. This is confirmed by the following gauge:

$$G^S(z) = \begin{pmatrix} \frac{1+z}{\sqrt{z}} & 0 \\ \lambda\frac{1-z}{\sqrt{z}} & \frac{\sqrt{z}}{1+z} \end{pmatrix}$$

A computation gives

$$\xi^S \cdot G^S = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} \frac{dz}{2(z+1)^2}$$

which is regular at 0 and ∞.

- Catenoidal limit: $(r,s) = (0, 1/2)$ gives

$$\xi^C = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \frac{dz}{z}$$

which we call the catenoidal Delaunay potential. The corresponding solution is

$$\Phi^C(z) = \frac{1}{2\sqrt{z}} \begin{pmatrix} z+1 & z-1 \\ z-1 & z+1 \end{pmatrix}$$

which does not depend on λ, so the immersion degenerates into the point 0. A computation gives

$$N^C(z) = \frac{1}{1+|z|^2}\left(1-|z|^2, 2\operatorname{Im}(z), 2\operatorname{Re}(z)\right).$$

which is a conformal diffeomorphism from $\mathbb{C}\cup\{\infty\}$ to $\mathbb{S}^2$.

3.8. **Duality.** Let

$$K(\lambda) = \begin{pmatrix} 0 & i\lambda^{-1/2} \\ i\lambda^{1/2} & 0 \end{pmatrix}.$$

Definition 9. The dual potential of ξ is

$$\xi^\dagger = K\xi K^{-1} = \begin{pmatrix} -\alpha & \lambda^{-1}\gamma \\ \beta & \alpha \end{pmatrix}.$$

The Delaunay spherical and catenoidal potentials are dual to each other. Note that K is not a gauge. Duality transforms the immersion in the following explicit way. Let $\Phi^\dagger = K\Phi K^{-1}$ be the solution of $d\Phi^\dagger = \Phi^\dagger\xi^\dagger$ with initial condition $\Phi^\dagger(z_0) = K\Phi(z_0)K^{-1}$. The Iwasawa decomposition of $\Phi^\dagger$ is $F^\dagger = KFK^{-1}$ and $B^\dagger = KBK^{-1}$. The Sym-Bobenko formula gives:

$$f^\dagger(z) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\left[f(z) + N(z) - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

In other words, up to a rigid motion, the dual (branched) immersion $f^\dagger$ is the parallel surface at distance one to f.

4. Strategy

Fix a horizontal weighted graph Γ. Until Section 9, we do not assume that Γ is balanced nor has length-2 edges. Without loss of generality, we may assume (by rotating the graph Γ) that $u_{jk} \neq \pm 1$ for all $(j,k) \in E \cup R$. We denote $\overline{\mathbb{C}}$ the Riemann sphere $\mathbb{C}\cup\{\infty\}$. Take a copy of the Riemann sphere $\overline{\mathbb{C}}_j$ for each $j \in J$, and a copy of the Riemann sphere $\overline{\mathbb{C}}_{jk}$ for each $(j,k) \in E^+$. For each $(j,k) \in E^+$, identify the point $z = u_{jk}$ in $\overline{\mathbb{C}}_j$ with the point $z = 1$ in $\overline{\mathbb{C}}_{jk}$, and the point $z = u_{kj}$ in $\overline{\mathbb{C}}_k$ with the point $z = -1$ in $\overline{\mathbb{C}}_{jk}$. This defines a compact Riemann surface with nodes $\overline{\Sigma}_0$ (the nodes are the double points created when identifying pairs of points).

Consider the meromorphic DPW potential ξ_0 on $\overline{\Sigma}_0$ defined by $\xi_0 = \xi^S$ in $\overline{\mathbb{C}}_j$ for $j \in J$ and $\xi_0 = \xi^C$ in $\overline{\mathbb{C}}_{jk}$ for $(j,k) \in E^+$. Fix an arbitrary $j_0 \in J$ and take as base point z_0 the point $z = 1$ in $\overline{\mathbb{C}}_{j_0}$ and the initial condition $\Phi(z_0) = I_2$. The fundamental group $\pi_1(\overline{\Sigma}_0, z_0)$ is generated by paths made of unit circular arcs connecting the nodes. Whenever a path γ crosses a node, we require the fundamental solution $\mathcal{P}(\xi_0, \gamma)$ to be continuous at the node. (This seems natural and is justified by the theoretical results of Appendix B: see Remark 14.)

The spherical and catenoidal potentials both take value in $\Lambda\mathfrak{su}(2)$ when $z \in \mathbb{S}^1$. So if all points u_{jk} are on the unit circle, the fundamental solution $\mathcal{P}(\xi_0, \gamma)$ will be in $\Lambda SU(2)$ for all $\gamma \in \pi_1(\overline{\Sigma}_0, z_0)$. Unitarization is the hard task in solving the Monodromy Problem, so this explains why we restrict to horizontal planar graphs Γ.

The strategy of the construction is the following: for small $t \neq 0$, we define a genuine Riemann surface $\overline{\Sigma}_t$ by opening the nodes of $\overline{\Sigma}_0$. We define a meromorphic potential ξ_t on $\overline{\Sigma}_t$ as a perturbation of the above potential ξ_0, depending on some parameters. These parameters are determined by solving the Regularity and Monodromy Problems by an implicit function argument at $t = 0$.

4.1. **Symmetry.** In all the paper, $\sigma(z) = 1/\overline{z}$ denotes the inversion with respect to the unit circle. The potentials ξ^S and ξ^C both have the symmetry

$$\overline{\sigma^*\xi} = D\xi D^{-1} \quad \text{with} \quad D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{12}$$

A potential having the symmetry (12) will be called σ-symmetric. With appropriate initial condition, the solution of the Cauchy Problem (3) satisfies

$$\overline{\sigma^*\Phi} = D\Phi D^{-1}. \tag{13}$$

The corresponding surface is invariant by the isometry $X \mapsto D\overline{X}D^{-1}$ in the $\mathfrak{su}(2)$-model, which corresponds to the symmetry with respect to the plane $x_1 = 0$. We keep the σ-symmetry throughout the construction, and in the end apply the rigid motion Ψ so that the surface is symmetric with respect to the horizontal plane $x_3 = 0$.

5. Opening nodes

In this section, we define a family of Riemann surfaces $\overline{\Sigma}_{t,\mathrm{x}}$ depending on a small real parameter t and a certain number of other parameters, which we denote x. We start by defining the Riemann surface with nodes $\overline{\Sigma}_{0,\mathrm{x}}$. We proceed as in Section 4 except that the position of the nodes in $\overline{\mathbb{C}}_j$ become parameters. (We can fix the nodes at 1 and -1 in $\overline{\mathbb{C}}_{jk}$ by a Möbius transformation.) Consider a copy $\overline{\mathbb{C}}_j$ of the Riemann sphere for $j \in J$ and a copy $\overline{\mathbb{C}}_{jk}$ of the Riemann sphere for $(j,k) \in E^+$. For $(j,k) \in E^+$, introduce two complex parameters p_{jk} and p_{kj} in a neighborhood of respectively u_{jk} and u_{kj}. It will be convenient to denote $p'_{jk} = 1$ and $p'_{kj} = -1$ the nodes in $\overline{\mathbb{C}}_{jk}$. Identify the point $z = p_{jk}$ in $\overline{\mathbb{C}}_j$ with the point $z = p'_{jk}$ in $\overline{\mathbb{C}}_{jk}$ and the point $z = p_{kj}$ in $\overline{\mathbb{C}}_k$ with the point $z = p'_{kj}$ in $\overline{\mathbb{C}}_{jk}$ to create two nodes per edge. This defines a compact Riemann surface with nodes denoted $\overline{\Sigma}_{0,\mathrm{x}}$.

To open nodes for $t \neq 0$, we introduce local complex coordinates in a neighborhood of p_{jk} and p'_{jk} for $(j,k) \in E$:

$$z_{jk} = -2i\frac{z - p_{jk}}{z + p_{jk}} : V_{jk} \subset \mathbb{C}_j \xrightarrow{\sim} D(0, \varepsilon).$$

$$z'_{jk} = -2i\frac{z - p'_{jk}}{z + p'_{jk}} : V'_{jk} \subset \overline{\mathbb{C}}_{jk} \xrightarrow{\sim} D(0, \varepsilon).$$

(These coordinates are chosen so that $\overline{\Sigma}_{t,\mathrm{x}}$ has the desired symmetry: see Proposition 1.) We assume that $\varepsilon > 0$ is small enough so that the disks V_{jk} for $k \in E_j$ are disjoint. For $(j,k) \in E$, we introduce a non-zero real parameter r_{jk} in a neighborhood of τ_{jk} and set $t_{jk} = r_{jk}t$. Assume that t is small enough so that $|t_{jk}| < \varepsilon^2$. Remove the disks $|z_{jk}| \leq |t_{jk}|/\varepsilon$ and $|z'_{jk}| \leq |t_{jk}|/\varepsilon$. Identify each point z in the annulus $|t_{jk}|/\varepsilon < |z_{jk}| < \varepsilon$ with the point z' in the annulus $|t_{jk}|/\varepsilon < |z'_{jk}| < \varepsilon$ such that

$$z_{jk}(z) z'_{jk}(z') = t_{jk}.$$

In particular, the circle $|z_{jk}| = |t_{jk}|^{1/2}$ is identified with the circle $|z'_{jk}| = |t_{jk}|^{1/2}$, with the reverse orientation. This creates two necks per edge. The resulting compact Riemann surface is denoted $\overline{\Sigma}_{t,\mathrm{x}}$. Note that it does not depend on λ. The points $z = 0$, $z = 1$ and $z = \infty$ in $\overline{\mathbb{C}}_j$ are denoted respectively 0_j, 1_j and ∞_j. The points $z = 0$ and $z = \infty$ in $\overline{\mathbb{C}}_{jk}$ are denoted 0_{jk} and ∞_{jk}.

Remark 3. The Riemann surface $\overline{\Sigma}_{t,\mathrm{x}}$ does not depend on the number $\varepsilon > 0$ used to define the domains V_{jk}, but the smaller ε, the smaller t must be since we need $|t_{jk}| < \varepsilon^2$.

5.1. Symmetry.

Proposition 1. *Assume that $p_{jk} \in \mathbb{S}^1$ for all $(j,k) \in E$. Then $\overline{\Sigma}_{t,\mathrm{x}}$ admits an anti-holomorphic involution σ defined by $\sigma(z) = 1/\overline{z}$ in $\overline{\mathbb{C}}_j$ for $j \in J$ and $\overline{\mathbb{C}}_{jk}$ for $(j,k) \in E^+$.*

Proof: a straightforward computation gives, for $p_{jk} \in \mathbb{S}^1$

$$\overline{z_{jk}(1/\overline{z})} = z_{jk}(z).$$

A similar relation holds for z'_{jk}. Hence since t_{jk} is real,

$$z_{jk}(z) z'_{jk}(z') = t_{jk} \quad \Rightarrow \quad z_{jk}(\sigma(z)) z'_{jk}(\sigma(z')) = t_{jk}.$$

So if $z \sim z'$ in $\overline{\Sigma}_{t,\mathrm{x}}$, then $\sigma(z) \sim \sigma(z')$ in $\overline{\Sigma}_{t,\mathrm{x}}$. □

5.2. Meromorphic 1-forms on $\overline{\Sigma}_{t,\mathrm{x}}$.

We denote $C(p_{jk})$ the circle $|z_{jk}| = \varepsilon$ and $C(p'_{jk})$ the circle $|z'_{jk}| = \varepsilon$. Assume $t \neq 0$ and let ω be a meromorphic 1-form on $\overline{\Sigma}_{t,\mathrm{x}}$ with poles outside of the annuli $|t_{jk}|/\varepsilon < |z_{jk}| < \varepsilon$. We have

$$\int_{C(p_{jk})} \omega = -\int_{C(p'_{jk})} \omega \quad \text{for } (j,k) \in E \tag{14}$$

because $C(p_{jk})$ is homologous to $-C(p'_{jk})$ in $\overline{\Sigma}_{t,\mathrm{x}}$. By the Residue Theorem in $\mathbb{C}_j$

$$\sum_{k \in E_j} \int_{C(p_{jk})} \omega + 2\pi i \sum_{q \in \mathbb{C}_j} \mathrm{Res}_q \omega = 0 \quad \text{for } j \in J \tag{15}$$

where the sum is taken on all poles q of ω in $\mathbb{C}_j$. In the same way,

$$\int_{C(p'_{jk})} \omega + \int_{C(p'_{kj})} \omega + 2\pi i \sum_{q \in \mathbb{C}_{jk}} \mathrm{Res}_q \omega = 0 \quad \text{for } (j,k) \in E^+. \tag{16}$$

Definition 10 (Bers)**.** A regular differential on the Riemann surface with nodes $\overline{\Sigma}_{0,\mathrm{x}}$ is a meromorphic 1-form with simple poles at the nodes p_{jk} and p'_{jk} for $(j,k) \in E$, with opposite residues, and possibly poles of arbitrary order away from the nodes.

Theorem 3. *A meromorphic 1-form ω on $\overline{\Sigma}_{t\neq 0,\mathrm{x}}$ (respectively a regular differential ω on $\overline{\Sigma}_{0,\mathrm{x}}$) is uniquely defined by prescribing its poles, principal parts at the poles and periods on the circles $C(p_{jk})$ and $C(p'_{jk})$ for $(j,k) \in E$, subject only to the constraints* (14), (15) *and* (16)*. Moreover, away from the nodes and the poles, ω depends holomorphically on t in a neighborhood of* 0 *and all parameters in the construction.*

This is proved for holomorphic 1-forms in [7] and for meromorphic 1-forms with simple poles in [17] using algebraic-geometric methods. A proof for poles of arbitrary order is given in [26]. The holomorphic dependence away from the nodes and the poles means the following: for $\epsilon > 0$, let Ω_ϵ be $\overline{\Sigma}_{0,\mathrm{x}}$ minus ϵ-neighborhoods of all nodes and poles, so $\Omega_\epsilon \subset \overline{\Sigma}_{t,\mathrm{x}}$ for t small enough. Then the restriction of ω to the fixed domain Ω_ϵ depends holomorphically on (z,t,x).

6. The potential

In this section, we define a meromorphic potential $\xi_{t,\mathrm{x}}$ on $\overline{\Sigma}_{t,\mathrm{x}}$, with poles at the following points:

- 0_j and ∞_j in $\overline{\mathbb{C}}_j$ for $j \in J$, which are to be apparent singularities,
- p_{jk} in $\overline{\mathbb{C}}_j$ for $(j,k) \in R$, which are to be the Delaunay ends of our surface. Here p_{jk} is a λ-dependent parameter in the functional space $\mathcal{W}^{\geq 0}$ in a neighborhood of u_{jk}, for $(j,k) \in R$.
- q_{jk} and $\sigma(q_{jk})$ in $\overline{\mathbb{C}}_{jk}$, for $(j,k) \in E^+$, which are to be apparent singularities. Here q_{jk} is a λ-dependent parameter in $\mathcal{W}^{\geq 0}$ in a neighborhood of 0, for $(j,k) \in E^+$.

Remark 4. All these λ-dependent parameters will be used to solve the Monodromy Problem. The cross-ratio of 1, -1, q_{jk} and $\sigma(q_{jk})$ is

$$(1,-1;q_{jk},\sigma(q_{jk})) = \frac{q_{jk}\overline{q_{jk}} - 1 + 2i\,\mathrm{Im}(q_{jk})}{1 - q_{jk}\overline{q_{jk}} + 2i\,\mathrm{Im}(q_{jk})}.$$

The derivative of the cross-ration with respect to $\mathrm{Re}(q_{jk})$ at $q_{jk} = 0$ is zero, so $\mathrm{Re}(q_{jk})$ serves no purpose and we restrict q_{jk} to the space $i\mathcal{W}^{\geq 0}_{\mathbb{R}}$. We could have fixed the singularities at 0_{jk} and ∞_{jk} and perturbed the position of the nodes at 1 and -1, but then $\overline{\Sigma}_{t,\mathrm{x}}$ would depend on λ. We chose to have a constant Riemann surface and moving singularities (with respect to λ), which is more conventional than the reverse.

We define the meromorphic potential $\xi_{t,\mathrm{x}}$ on $\overline{\Sigma}_{t,\mathrm{x}}$ as the sum of two terms:

$$\xi_{t,\mathrm{x}} = \eta_{t,\mathrm{x}} + t\,\chi_{t,\mathrm{x}}$$

where the potential $\eta_{t,\mathrm{x}}$ is a perturbation of the potential ξ_0 described in Section 4, while the potential $\chi_{t,\mathrm{x}}$ prescribes periods around the nodes and suitable singularities at the Delaunay ends. These potentials are defined as follows, using Theorem 3:

- The potential $\eta_{t,\mathrm{x}}$ has simple poles at 0_j and ∞_j for $j \in J$ with residues

$$\mathrm{Res}_{0_j}\eta_{t,\mathrm{x}} = -\mathrm{Res}_{\infty_j}\eta_{t,\mathrm{x}} = M_j = \begin{pmatrix} iA_j & \lambda^{-1}B_j \\ \lambda C_j & -iA_j \end{pmatrix},$$

  simple poles at q_{jk} and $\sigma(q_{jk})$ for $(j,k) \in E^+$ with residues

$$\mathrm{Res}_{q_{jk}}\eta_{t,\mathrm{x}} = -\mathrm{Res}_{\sigma(q_{jk})}\eta_{t,\mathrm{x}} = M_{jk} = \begin{pmatrix} iA_{jk} & B_{jk} \\ C_{jk} & -iA_{jk} \end{pmatrix}$$

  and has vanishing periods around the nodes:

$$\int_{C(p_{jk})} \eta_{t,\mathrm{x}} = \int_{C(p'_{jk})} \eta_{t,\mathrm{x}} = 0 \quad \text{for } (j,k) \in E.$$

  Here A_j, B_j, C_j, A_{jk}, B_{jk}, C_{jk} are parameters in a neighborhood of respectively 0, 1/2, 1/2, 0, 1/2, 1/2 in $\mathcal{W}^{\geq 0}_{\mathbb{R}}$.
- The potential $\chi_{t,\mathrm{x}}$ has the following periods around the nodes for $(j,k) \in E$:

$$\int_{C(p_{jk})} \chi_{t,\mathrm{x}} = -\int_{C(p'_{jk})} \chi_{t,\mathrm{x}} = 2\pi i\, m_{jk} \quad \text{with} \quad m_{jk} = \begin{pmatrix} a_{jk} & \lambda^{-1} i b_{jk} \\ i c_{jk} & -a_{jk} \end{pmatrix}$$

where a_{jk}, b_{jk}, c_{jk} for $(j,k)\in E$ are parameters in $\mathcal{W}_{\mathbb{R}}^{\geq 0}$ to be determined. It has a double pole at p_{jk} in $\mathbb{C}_j$ for $(j,k)\in R$ with principal part

$$\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}\left(\frac{a_{jk}p_{jk}dz}{(z-p_{jk})^2}+\frac{ib_{jk}dz}{z-p_{jk}}\right).$$

Here a_{jk}, b_{jk} are parameters in $\mathcal{W}_{\mathbb{R}}^{\geq 0}$ to be determined, for $(j,k)\in R$. It is known from [28] that such a pole creates a Delaunay end, provided the Monodromy Problem is solved. Finally, the potential $\chi_{t,\mathrm{x}}$ has simple poles with equal residues at 0_j and ∞_j and simples poles with equal residues at q_{jk} and $\sigma(q_{jk})$. These residues are determined by the constraints (15) and (16) which give:

$$\mathrm{Res}_{0_j}\chi_{t,\mathrm{x}}=\mathrm{Res}_{\infty_j}\chi_{t,\mathrm{x}}=-\frac{1}{2}\sum_{k\in E_j}m_{jk}-\frac{1}{2}\sum_{k\in R_j}\begin{pmatrix} 0 & 0\\ ib_{jk} & 0\end{pmatrix} \tag{17}$$

$$\mathrm{Res}_{q_{jk}}\chi_{t,\mathrm{x}}=\mathrm{Res}_{\sigma(q_{jk})}\chi_{t,\mathrm{x}}=\frac{1}{2}(m_{jk}+m_{kj}). \tag{18}$$

6.1. **Symmetry.** The residues and periods of the entries of $\eta_{t,\mathrm{x}}$ and $\chi_{t,\mathrm{x}}$ have been chosen to be either real or imaginary so that the potential has the desired symmetry:

Proposition 2. *Assume that $p_{jk}\in\mathbb{S}^1$ for $(j,k)\in E$ and $p_{jk}=e^{i\theta_{jk}}$ with $\theta_{jk}\in\mathcal{W}_{\mathbb{R}}^{\geq 0}$ for $(j,k)\in R$. Then the potential $\xi_{t,\mathrm{x}}$ has the symmetry* (12)*:*

$$\overline{\sigma^*\xi_{t,\mathrm{x}}}=D\xi_{t,\mathrm{x}}D^{-1}.$$

Note that the bar denotes the conjugation operator defined in Section 3.1 so this actually means $\overline{\sigma^*\xi_{t,\mathrm{x}}(z,\overline{\lambda})}=D\xi_{t,\mathrm{x}}(z,\lambda)D^{-1}$. Both sides are holomorphic with respect to λ.

Proof: if ω is a meromorphic 1-form on $\overline{\Sigma}_{t,\mathrm{x}}$ then $\overline{\sigma^*\omega}$ is meromorphic and

$$\mathrm{Res}_{\sigma(p)}\overline{\sigma^*\omega}=\overline{\mathrm{Res}_p\omega}.$$

Hence

$$\mathrm{Res}_{\infty_j}\overline{\sigma^*\eta_{t,\mathrm{x}}}=\overline{\mathrm{Res}_{0_j}\eta_{t,\mathrm{x}}}=\overline{M_j}=-DM_jD^{-1}=\mathrm{Res}_{\infty_j}D\eta_{t,\mathrm{x}}D^{-1}.$$

In the same way, $\overline{\sigma^*\eta_{t,\mathrm{x}}}$ and $D\eta_{t,\mathrm{x}}D^{-1}$ have the same residues at 0_j, q_{jk} and $\sigma(q_{jk})$. Moreover, both have vanishing periods around the nodes, so by uniqueness in Theorem 3,

$$\overline{\sigma^*\eta_{t,\mathrm{x}}}=D\eta_{t,\mathrm{x}}D^{-1}.$$

For $(j,k)\in E$, we have since $\sigma(C(p_{jk}))=-C(p_{jk})$

$$\int_{C(p_{jk})}\overline{\sigma^*\chi_{t,\mathrm{x}}}=-\int_{C(p_{jk})}\overline{\chi_{t,\mathrm{x}}}=2\pi i\,\overline{m_{jk}}=2\pi iDm_{jk}D^{-1}$$

so $\overline{\sigma^*\chi_{t,\mathrm{x}}}$ and $D\chi_{t,\mathrm{x}}D^{-1}$ have the same periods around the nodes. For $(j,k)\in R$, assuming that $p_{jk}=e^{i\theta_{jk}}$ with $\theta_{jk}\in\mathcal{W}_{\mathbb{R}}^{\geq 0}$, we have $\sigma(p_{jk})=p_{jk}$ and

$$\overline{\sigma^*\left(\frac{a_{jk}p_{jk}dz}{(z-p_{jk})^2}+\frac{ib_{jk}dz}{z-p_{jk}}\right)}=-\frac{a_{jk}p_{jk}dz}{(z-p_{jk})^2}-\frac{ib_{jk}dz}{z-p_{jk}}+\frac{ib_{jk}dz}{z}$$

so $\overline{\sigma^*\chi_{t,\mathrm{x}}}$ and $D\chi_{t,\mathrm{x}}D^{-1}$ have the same principal part at p_{jk}. Finally, they have the same residues at 0_j, ∞_j, q_{jk} and $\sigma(q_{jk})$ by computations similar to the above. By uniqueness in Theorem 3, we have

$$\overline{\sigma^*\chi_{t,\mathrm{x}}}=D\chi_{t,\mathrm{x}}D^{-1}.$$

$\square$

6.2. **Explicit formulas at $t = 0$.** It will be convenient to denote, for $(j,k) \in E^+$, $M_{kj} = M_{jk}$, $A_{kj} = A_{jk}$, etc... so M_{jk} makes sense for all $(j,k) \in E$. Be careful however that A_{jk} and A_{kj} are the same parameter, whereas a_{jk} and a_{kj} are distinct parameters. For complex numbers p, q, we denote ω_q the meromorphic 1-form on the Riemann sphere with simple poles at q and $\sigma(q)$ with residues 1 and -1, and $\omega_{p,q}$ the meromorphic 1-form with simple poles at p, q and $\sigma(q)$ with residues 1, $-1/2$ and $-1/2$. Explicitly:

$$\omega_q = \frac{dz}{z-q} - \frac{\overline{q}dz}{\overline{q}z-1} = \frac{(1-q\overline{q})dz}{(z-q)(1-\overline{q}z)} \quad \text{and} \quad \omega_{p,q} = \frac{dz}{z-p} - \frac{dz}{2(z-q)} - \frac{\overline{q}\,dz}{2(\overline{q}z-1)}.$$

In particular if $q = 0$:

$$\omega_0 = \frac{dz}{z} \quad \text{and} \quad \omega_{p,0} = \frac{dz}{z-p} - \frac{dz}{2z}.$$

Proposition 3. *At $t = 0$ and for any value of the parameter x, we have in $\overline{\mathbb{C}}_j$ for $j \in J$:*

$$\eta_{0,\mathrm{x}} = M_j \omega_0$$

$$\chi_{0,\mathrm{x}} = \sum_{k\in E_j} m_{jk}\omega_{p_{jk},0} + \sum_{k \in R_j} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \left(\frac{a_{jk}p_{jk}\,dz}{(z-p_{jk})^2} + ib_{jk}\omega_{p_{jk},0} \right)$$

and in $\overline{\mathbb{C}}_{jk}$ for $(j,k) \in E^+$:

$$\eta_{0,\mathrm{x}} = M_{jk}\omega_{q_{jk}}$$

$$\chi_{0,\mathrm{x}} = -m_{jk}\omega_{1,q_{jk}} - m_{kj}\omega_{-1,q_{jk}}.$$

Proof: the entries of $\eta_{0,\mathrm{x}}$ and $\chi_{0,\mathrm{x}}$ are regular meromorphic differentials on the Riemann surface with nodes $\overline{\Sigma}_{0,\mathrm{x}}$. Proposition 3 follows from the fact that a meromorphic 1-form on the Riemann sphere is uniquely defined by its poles and principal parts. □

We shall need the t-derivative of the potential $\xi_{t,\mathrm{x}}$ at $t = 0$. We have of course

$$\frac{\partial \xi_{t,\mathrm{x}}}{\partial t}\,|_{t=0} = \frac{\partial \eta_{t,\mathrm{x}}}{\partial t}\,|_{t=0} + \chi_{0,\mathrm{x}}.$$

Proposition 4. *The t-derivative of the potential $\eta_{t,\mathrm{x}}$ at $t = 0$ is given by*

$$\frac{\partial \eta_{t,\mathrm{x}}}{\partial t}\,|_{t=0} = \begin{cases} \displaystyle\sum_{k\in E_j} r_{jk}M_{jk}\frac{(1+q_{jk}^2)}{(1-q_{jk}^2)}\frac{p_{jk}dz}{(z-p_{jk})^2} & \text{in } \overline{\mathbb{C}}_j \text{ for } j \in J \\ \displaystyle r_{jk}M_j\frac{dz}{(z-1)^2} - r_{kj}M_k\frac{dz}{(z+1)^2} & \text{in } \overline{\mathbb{C}}_{jk} \text{ for } (j,k) \in E^+. \end{cases}$$

Proof: by Lemma 3 in [25], for $(j,k) \in E$, the derivative of $\eta_{t,\mathrm{x}}$ with respect to the parameter t_{jk} at $t = 0$ is a meromorphic differential on $\overline{\Sigma}_{0,\mathrm{x}}$ with two double poles at p_{jk}, p'_{jk} and principal parts given in term of the coordinates used to open nodes by

$$\frac{\partial \eta_{t,\mathrm{x}}}{\partial t_{jk}}\,|_{t=0} \simeq \begin{cases} \displaystyle\frac{-dz_{jk}}{(z_{jk})^2}\mathrm{Res}_{p'_{jk}}\frac{\eta_{0,\mathrm{x}}}{z'_{jk}} & \text{at } p_{jk} \\ \displaystyle\frac{-dz'_{jk}}{(z'_{jk})^2}\mathrm{Res}_{p_{jk}}\frac{\eta_{0,\mathrm{x}}}{z_{jk}} & \text{at } p'_{jk} \end{cases}. \tag{19}$$

We have

$$\frac{dz_{jk}}{(z_{jk})^2} = \frac{ip_{jk}\,dz}{(z-p_{jk})^2} \quad \text{and} \quad \frac{dz'_{jk}}{(z'_{jk})^2} = \frac{ip'_{jk}\,dz}{(z-p'_{jk})^2}.$$

Observe that these are globally defined meromorphic 1-forms on the Riemann sphere so $\simeq$ in (19) becomes an equality in $\overline{\mathbb{C}}_j$ and $\overline{\mathbb{C}}_{jk}$, respectively. By Proposition 3:

$$\operatorname{Res}_{p_{jk}} \frac{\eta_{0,\mathrm{x}}}{z_{jk}} = \operatorname{Res}_{p_{jk}} \frac{(z+p_{jk})}{-2i(z-p_{jk})} M_j \frac{dz}{z} = iM_j$$

Recalling that $q_{jk} \in i\mathcal{W}_{\mathbb{R}}^{\geq 0}$ so $\overline{q_{jk}} = -q_{jk}$ and that $p'_{jk} = \pm 1$:

$$\operatorname{Res}_{p'_{jk}} \frac{\eta_{0,\mathrm{x}}}{z'_{jk}} = \operatorname{Res}_{p'_{jk}} \frac{(z+p'_{jk})}{-2i(z-p'_{jk})} M_{jk} \frac{(1+q_{jk}^2)dz}{(z-q_{jk})(1+q_{jk}z)} = iM_{jk} \frac{1+q_{jk}^2}{1-q_{jk}^2}.$$

Hence for $(j,k) \in E$:

$$\frac{\partial \eta_{t,\mathrm{x}}}{\partial t_{jk}} \,|_{t=0} = \begin{cases} M_{jk} \dfrac{(1+q_{jk}^2)}{(1-q_{jk}^2)} \dfrac{p_{jk}\, dz}{(z-p_{jk})^2} & \text{in } \overline{\mathbb{C}}_j \\ M_j \dfrac{p'_{jk} dz}{(z-p'_{jk})^2} & \text{in } \overline{\mathbb{C}}_{jk} \\ 0 & \text{in all other Riemann spheres.} \end{cases}$$

Proposition 4 follows from $t_{jk} = r_{jk} t$ and the chain rule. □

6.3. **Central value of the parameters.** The vector of all parameters of the construction (except t) is denoted x. Each parameter is in a neighborhood of a central value denoted with an underscore. The central values are tabulated below. Some of them we have already seen. The others will be computed when solving the Monodromy Problem.

Also, we have tried to define the potential in a way as general and natural as possible, but it turns out *a posteriori* after solving all equations that we have too many parameters, so we can fix the value of some of them: A_j, B_j for $j \in J$ will not be used. Some computations are simpler with these restrictions so we assume them from now on.

parameter	range	space	central value
p_{jk}	$(j,k) \in E$	$\mathbb{S}^1$	u_{jk}
p'_{jk}	$(j,k) \in E$	fixed	± 1
r_{jk}	$(j,k) \in E$	$\mathbb{R}$	τ_{jk}
A_j	$j \in J$	fixed	0
B_j	$j \in J$	fixed	1/2
C_j	$j \in J$	$\mathcal{W}_{\mathbb{R}}^{\geq 0}$	1/2
q_{jk}	$(j,k) \in E^+$	$i\mathcal{W}_{\mathbb{R}}^{\geq 0}$	0
A_{jk}	$(j,k) \in E^+$	$\mathcal{W}_{\mathbb{R}}^{\geq 0}$	0
B_{jk}, C_{jk}	$(j,k) \in E^+$	$\mathcal{W}_{\mathbb{R}}^{\geq 0}$	1/2
a_{jk}	$(j,k) \in E$	$\mathcal{W}_{\mathbb{R}}^{\geq 0}$	$\tau_{jk}(\lambda-1)/2$
b_{jk}, c_{jk}	$(j,k) \in E$	$\mathcal{W}_{\mathbb{R}}^{\geq 0}$	0
p_{jk}	$(j,k) \in R$	$\exp(i\mathcal{W}_{\mathbb{R}}^{\geq 0})$	u_{jk}
a_{jk}	$(j,k) \in R$	$\mathcal{W}_{\mathbb{R}}^{\geq 0}$	$\tau_{jk}(\lambda-1)^2/2$
b_{jk}	$(j,k) \in R$	$\mathcal{W}_{\mathbb{R}}^{\geq 0}$	0

7. The Regularity Problem

We want 0_j, ∞_j and q_{jk}, $\sigma(q_{jk})$ to be apparent singularities. In this section, the entries of the potential will be denoted

$$\xi_{t,\mathrm{x}} = \begin{pmatrix} \alpha & \lambda^{-1}\beta \\ \gamma & -\alpha \end{pmatrix} \tag{20}$$

and the dependence on the parameters (t, x) will not be written to ease notations.

7.1. **Regularity at 0_j and ∞_j.** Fix $j \in J$ and consider the gauge

$$G_j = \begin{pmatrix} f & 0 \\ \lambda g & f^{-1} \end{pmatrix} \quad \text{with} \quad f(z) = \frac{1+z}{\sqrt{z}} \quad \text{and} \quad g(z) = x_j \frac{1-z}{\sqrt{z}} + i\, y_j \frac{1+z}{\sqrt{z}}.$$

Here x_j, y_j are parameters in $\mathcal{W}_{\mathbb{R}}^{\geq 0}$ to be determined. At $(x_j, y_j) = (1, 0)$ we have $G_j = G^S$. We denote

$$\widehat{\xi} = \xi_{t,\mathrm{x}} \cdot G_j = \begin{pmatrix} \widehat{\alpha} & \lambda^{-1}\widehat{\beta} \\ \widehat{\gamma} & -\widehat{\alpha} \end{pmatrix}.$$

The gauge has the symmetry $\overline{G_j \circ \sigma} = D G_j D^{-1}$ so $\widehat{\xi}$ has the symmetry (12) and it suffices to ensure that $\widehat{\xi}$ is regular at 0_j ; regularity at ∞_j will follow by symmetry.

Proposition 5. *There exists explicit values of x_j, y_j and C_j in $\mathcal{W}_{\mathbb{R}}^{\geq 0}$, depending analytically on (t, x), such that $\widehat{\alpha}$ and $\widehat{\beta}$ are holomorphic at 0_j, $\widehat{\gamma}$ has a pole of multiplicity at most 2 and*

$$\operatorname{Re}\left(\operatorname{Res}_{0_j}(z\widehat{\gamma})\right) = 0. \tag{21}$$

Proof: straightforward computations give

$$\begin{aligned} \widehat{\alpha} &= \alpha + f^{-1} g \beta + f^{-1} df \\ \widehat{\beta} &= f^{-2}\beta \\ \widehat{\gamma} &= -2\lambda f g \alpha - \lambda g^2 \beta + f^2 \gamma + \lambda(f\, dg - g\, df). \end{aligned} \tag{22}$$

Recall that α, β, γ have simple poles at 0_j. Hence $\widehat{\beta}$ is holomorphic at 0_j and $\widehat{\alpha}$ has (at most) a simple pole with residue

$$\operatorname{Res}_{0_j}\widehat{\alpha} = \operatorname{Res}_{0_j}\alpha + (x_j + i y_j)\operatorname{Res}_{0_j}\beta - \frac{1}{2}.$$

We take

$$x_j + i y_j = \frac{1/2 - \operatorname{Res}_{0_j}\alpha}{\operatorname{Res}_{0_j}\beta} \tag{23}$$

so that $\widehat{\alpha}$ is holomorphic at 0_j. Finally, $\widehat{\gamma}$ has at most a double pole at 0_j and since $f\, dg - g\, df$ has a simple pole at 0,

$$\operatorname{Res}_{0_j}(z\widehat{\gamma}) = -2\lambda(x_j + i y_j)\operatorname{Res}_{0_j}\alpha - \lambda(x_j + i y_j)^2 \operatorname{Res}_{0_j}\beta + \operatorname{Res}_{0_j}\gamma.$$

By definition, recalling the definition of the operator Re in Section 3.1:

$$\operatorname{Re}(\operatorname{Res}_{0_j}\gamma) = \lambda C_j.$$

So we see that Equation (21) is equivalent to

$$C_j = \operatorname{Re}\left[2(x_j + i y_j)\operatorname{Res}_{0_j}\alpha + (x_j + i y_j)^2 \operatorname{Res}_{0_j}\beta\right]$$

which using Equation (23) simplifies to

$$C_j = \operatorname{Re}\left(\frac{1/4 - (\operatorname{Res}_{0_j}\alpha)^2)}{\operatorname{Res}_{0_j}\beta}\right). \tag{24}$$

Note that the residues of α and β involved in Equations (23) and (24) are given, as functions of (t, x), by the definition of $\xi_{t,\mathrm{x}}$. In particular, at $t = 0$, we have $x_j = 1$ and $y_j = 0$ so $G_j = G^S$, and

$$C_j \mid_{t=0} = \frac{1/4 + A_j^2}{B_j} = 1/2. \tag{25}$$

□

At this point, the Regularity Problem at 0_j is only partially solved since $\widehat{\gamma}$ still has a pole. By Equation (22), we have

$$\widehat{\gamma}^0 = z^{-1}(z+1)^2\gamma^0 \tag{26}$$

so for $\widehat{\gamma}$ to be holomorphic at 0_j, it is necessary that

$$\text{Res}_{0_j}\left(z^{-1}(z+1)^2\gamma^0\right) = 0.$$

We define for $j \in J$ and $t \neq 0$

$$\mathcal{R}_j(t,\text{x}) = t^{-1}\text{Res}_{0_j}\left(z^{-1}(z+1)^2\gamma^0_{t,\text{x}}\right) \in \mathbb{C}. \tag{27}$$

Proposition 6. *For $j \in J$, the function $\mathcal{R}_j(t,\text{x})$ extends analytically at $t=0$. Moreover, at the central value, we have $\mathcal{R}_j(0,\underline{\text{x}}) = \overline{F_j}/2$, where F_j is the force defined in Equation (1).*

Proof: by Proposition 3, we have $\gamma_{0,\text{x}} = \lambda C_j\omega_0$ in $\overline{\mathbb{C}}_j$ so $\gamma^0_{0,\text{x}} = 0$. Hence $\mathcal{R}_j$ extends analytically at $t=0$ and

$$\mathcal{R}_j(0,\text{x}) = \text{Res}_{0_j}\left(z^{-1}(z+1)^2\frac{\partial\gamma^0_{t,\text{x}}}{\partial t}\,|_{t=0}\right).$$

By Proposition 4, we have

$$\frac{\partial\gamma_{t,\text{x}}}{\partial t}\,|_{t=0} = \sum_{k\in E_j}\left(r_{jk}C_{jk}\frac{(1+q_{jk}^2)}{(1-q_{jk}^2)}\frac{p_{jk}\,dz}{(z-p_{jk})^2} + ic_{jk}\omega_{p_{jk},0}\right) + \sum_{k\in R_j}\left(\frac{a_{jk}p_{jk}\,dz}{(z-p_{jk})^2} + ib_{jk}\omega_{p_{jk},0}\right).$$

At the central value (see the table in Section 6.3) and $\lambda = 0$, this simplifies to

$$\frac{\partial\gamma^0_{t,\underline{\text{x}}}}{\partial t}\,|_{t=0} = \sum_{k\in E_j\cup R_j}\frac{\tau_{jk}u_{jk}\,dz}{2(z-u_{jk})^2}$$

which is holomorphic at 0_j. Hence

$$\mathcal{R}_j(0,\underline{\text{x}}) = \sum_{k\in E_j\cup R_j}\frac{\tau_{jk}}{2u_{jk}}.$$

□

Remark 5. Proposition 6 explains where the balancing condition comes from. We solve the equation $\mathcal{R}_j = 0$ in Section 9 using the non-degeneracy hypothesis. Then after the Monodromy Problem is solved, $\widehat{\gamma}$ will in fact be holomorphic at 0_j: see Proposition 18.

7.2. **Regularity at q_{jk} and $\sigma(q_{jk})$.** Fix $(j,k) \in E^+$. Recall that $\xi_{t,\text{x}}$ has moving singularities at q_{jk} and $\sigma(q_{jk})$, which depend on λ. We use the following Möbius transformation as local coordinate in a neighborhood of q_{jk}:

$$w_{jk}(z) = \frac{z-q_{jk}}{1-\overline{q_{jk}}z} = \frac{z-q_{jk}}{1+q_{jk}z}$$

We have $\sigma\circ w_{jk} = w_{jk}\circ\sigma$. We make the change of variable $w = w_{jk}$ and denote

$$\widetilde{\xi} = (w_{jk}^{-1})^*\xi_{t,\text{x}} = \begin{pmatrix}\widetilde{\alpha} & \lambda^{-1}\widetilde{\beta}\\ \widetilde{\gamma} & -\widetilde{\alpha}\end{pmatrix}$$

which has fixed singularities at $w=0$ and $w=\infty$ and still has the symmetry (12). We consider a gauge G_{jk} of a form dual to G_j:

$$G_{jk} = \begin{pmatrix} f^{-1} & g\\ 0 & f\end{pmatrix} \quad\text{with}\quad f = \frac{1+w}{\sqrt{w}} \quad\text{and}\quad g = x_{jk}\frac{1-w}{\sqrt{w}} + i\,y_{jk}\frac{1+w}{\sqrt{w}}.$$

Let

$$\widehat{\xi} = \widetilde{\xi} \cdot G_{jk} = \begin{pmatrix} \widehat{\alpha} & \lambda^{-1}\widehat{\beta} \\ \widehat{\gamma} & -\widehat{\alpha} \end{pmatrix}.$$

The gauge G_{jk} has the symmetry $\overline{G_{jk} \circ \sigma} = DG_{jk}D^{-1}$ so it suffices to ensure that $\widehat{\xi}$ is regular at $w = 0$, regularity at $w = \infty$ will follow by symmetry.

Proposition 7. *There exists explicit values of x_{jk}, y_{jk} and B_{jk} in $\mathcal{W}_{\mathbb{R}}^{\geq 0}$, depending analytically on (t, x), such that $\widehat{\alpha}$ and $\widehat{\gamma}$ are holomorphic at $w = 0$, $\widehat{\beta}$ has a pole of multiplicity at most 2 and*

$$\mathrm{Re}\left(\mathrm{Res}_0(w\widehat{\beta})\right) = 0. \tag{28}$$

Proof: we simply dualize the proof of Proposition 5 with $\widetilde{\xi}$ in place of $\xi_{t,\mathrm{x}}$ and obtain:

$$x_{jk} + iy_{jk} = \frac{1/2 + \mathrm{Res}_0\widetilde{\alpha}}{\mathrm{Res}_0\widetilde{\gamma}} = \frac{1/2 + \mathrm{Res}_{q_{jk}}\alpha}{\mathrm{Res}_{q_{jk}}\gamma} \tag{29}$$

$$B_{jk} = \mathrm{Re}\left(\frac{1/4 - (\mathrm{Res}_0\widetilde{\alpha})^2}{\mathrm{Res}_0\widetilde{\gamma}}\right) = \mathrm{Re}\left(\frac{1/4 - (\mathrm{Res}_{q_{jk}}\alpha)^2}{\mathrm{Res}_{q_{jk}}\gamma}\right) \tag{30}$$

$$B_{jk}\mid_{t=0} = \frac{1/4 + A_{jk}^2}{C_{jk}}. \tag{31}$$

□

At this point, the Regularity Problem at q_{jk} is only partially solved since $\widehat{\beta}$ still has a pole. Dualizing Equation (26) we have

$$\widehat{\beta}^0 = w^{-1}(w+1)^2\widetilde{\beta}^0 \tag{32}$$

For $\widehat{\beta}$ to be holomorphic, it is necessary that

$$\mathrm{Res}_0\left(w^{-1}(w+1)^2\widetilde{\beta}^0\right) = 0.$$

We define for $(j,k) \in E^+$ and $t \neq 0$:

$$\mathcal{R}_{jk}(t,\mathrm{x}) = t^{-1}\mathrm{Res}_0\left(w^{-1}(w+1)^2\widetilde{\beta}^0\right) \in \mathbb{C}.$$

Proposition 8. *For $(j,k) \in E^+$, the function $\mathcal{R}_{jk}$ extends analytically at $t = 0$ and*

$$\mathcal{R}_{jk}(0,\mathrm{x}) = r_{jk}\frac{1 + (q_{jk}^0)^2}{2(1 - q_{jk}^0)^2} - r_{kj}\frac{1 + (q_{jk}^0)^2}{2(1 + q_{jk}^0)^2} + \frac{2ib_{jk}^0}{1 - q_{jk}^0} + \frac{2iq_{jk}^0 b_{kj}^0}{1 + q_{jk}^0}. \tag{33}$$

In particular, $\mathcal{R}_{jk}(0,\underline{\mathrm{x}}) = 0$ at the central value.

Proof: by Proposition 3, we have $\beta_{0,\mathrm{x}} = \lambda B_{jk}\omega_{q_{jk}}$ in $\overline{\mathbb{C}}_{jk}$ so $\widetilde{\beta}_{0,\mathrm{x}}^0 = 0$. Hence $\mathcal{R}_{jk}$ extends analytically at $t = 0$ and

$$\mathcal{R}_{jk}(0,\mathrm{x}) = \mathrm{Res}_0\left(w^{-1}(w+1)^2\frac{\partial\widetilde{\beta}_{t,\mathrm{x}}^0}{\partial t}\mid_{t=0}\right).$$

By Proposition 4, remembering that we fixed $B_j = B_k = 1/2$:

$$\frac{\partial\beta_{t,\mathrm{x}}}{\partial t}\mid_{t=0} = \frac{r_{jk}\,dz}{2(z-1)^2} - \frac{r_{kj}\,dz}{2(z+1)^2} - ib_{jk}\omega_{1,q_{jk}} - ib_{kj}\omega_{-1,q_{kj}}.$$

The first two residues are better computed using the z-coordinate

$$\mathrm{Res}_{w=0}\left(w^{-1}(w+1)^2(w_{jk}^{-1})^*\frac{dz}{(z\pm1)^2}\right) = \mathrm{Res}_{z=q_{jk}}\left(w_{jk}^{-1}(w_{jk}+1)^2\frac{dz}{(z\pm1)^2}\right) = \frac{1+q_{jk}^2}{(q_{jk}\pm1)^2}.$$

The last two residues are better computed using the w-coordinate:

$$(w_{jk}^{-1})^*\omega_{\pm 1,q_{jk}} = \frac{dw}{w - w_{jk}(\pm 1)} - \frac{dw}{2w}$$

$$\text{Res}_0\left(w^{-1}(w+1)^2(w_{jk}^{-1})^*\omega_{1,q_{jk}}\right) = \frac{-1}{w_{jk}(1)} - 1 = \frac{2}{q_{jk}-1}$$

$$\text{Res}_0\left(w^{-1}(w+1)^2(w_{jk}^{-1})^*\omega_{-1,q_{jk}}\right) = \frac{-1}{w_{jk}(-1)} - 1 = \frac{-2q_{jk}}{q_{jk}+1}.$$

Collecting all terms and setting $\lambda = 0$, we obtain Equation (33). □

Remark 6. We solve the equation $\mathcal{R}_{jk}(t, \mathrm{x}) = 0$ using the Implicit Function Theorem in Section 9. Then after the Monodromy Problem is solved, $\widehat{\beta}$ will in fact be holomorphic at $w = 0$: see Proposition 20.

8. The Monodromy Problem

From now on, we assume that C_j is given in function of (t, x) by Equation (24) for $j \in J$ and B_{jk} is given by Equation (30) for $(j,k) \in E^+$. Also, we restrict t to be positive.

8.1. **Definition of various paths.** In this section, we define for $(j,k) \in E \cup R$ a loop γ_{jk} with base point 1_j encircling the point p_{jk}, and for $(j,k) \in E^+$ a path Γ_{jk} connecting 1_j to 1_k through the two necks corresponding to the edge (j,k) (see Figure 2). We study carefully how these paths transform under σ.

Fix $j \in J$. We define an order $\prec$ on the set $E_j \cup R_j$ by $k \prec \ell \Leftrightarrow \arg(u_{jk}) < \arg(u_{j\ell})$, where the arguments are chosen in $(0, 2\pi)$. For $k \in E_j \cup R_j$, we fix a curve α_{jk} in the domain $\{z \in \mathbb{C}_j : |z| > 1, 0 < \arg(z) < 2\pi\}$ from 1_j to $e^{i\varepsilon}u_{jk}$ and define $\delta_{jk} = \alpha_{jk}\sigma(\alpha_{jk})^{-1}$. The domain bounded by δ_{jk} contains the points $p_{j\ell}$ for $\ell \preceq k$. We define inductively the loops γ_{jk} for $k \in E_j \cup R_j$ by

$$\delta_{jk} = \prod_{\ell \preceq k} \gamma_{j\ell}. \tag{34}$$

In other words, $\gamma_{jk} = (\delta_{jk'})^{-1}\delta_{jk}$ where k' is the predecessor of k for the order $\prec$. The domain bounded by γ_{jk} contains the point p_{jk} and no other $p_{j\ell}$. It will be convenient to also denote

$$\delta'_{jk} = \prod_{\ell \prec k} \gamma_{j\ell}$$

so $\delta_{jk} = \delta'_{jk}\gamma_{jk}$. (An empty product means the neutral element.) These paths transform as follows under σ:

$$\sigma(\delta_{jk}) = \delta_{jk}^{-1} \tag{35}$$

$$\sigma(\gamma_{jk}) = \delta'_{jk}\gamma_{jk}^{-1}(\delta'_{jk})^{-1}. \tag{36}$$

Fix $(j,k) \in E^+$. The path Γ_{jk} is defined as follows. Fix a number ε' such that $0 < \varepsilon' < \varepsilon$, where ε is the number introduced to open nodes in section 5. Recalling the definition of the coordinate z_{jk} near p_{jk}, we have

$$z = p_{jk}\frac{(2 + iz_{jk})}{(2 - iz_{jk})}$$

so for real $x \in [-\varepsilon, \varepsilon]$, the point $z_{jk} = x$ is on the unit circle and its argument is an increasing function of x. First assume that $\tau_{jk} > 0$ so t_{jk} and t_{kj} are positive. We define the path β_{jk} as the concatenation of the following 5 paths (taking care to avoid the disks that are removed when opening nodes):

(1) The circular arc from $z = e^{i\varepsilon}u_{jk}$ to $z_{jk} = \varepsilon'$.
(2) The circular arc from $z_{jk} = \varepsilon'$ to $z_{jk} = t_{jk}/\varepsilon'$. Its endpoint was identified with $z'_{jk} = \varepsilon'$ when opening nodes.
(3) The circular arc from $z'_{jk} = \varepsilon'$ to $z'_{kj} = -\varepsilon'$ on the upper half unit circle in $\overline{\mathbb{C}}_{jk}$.

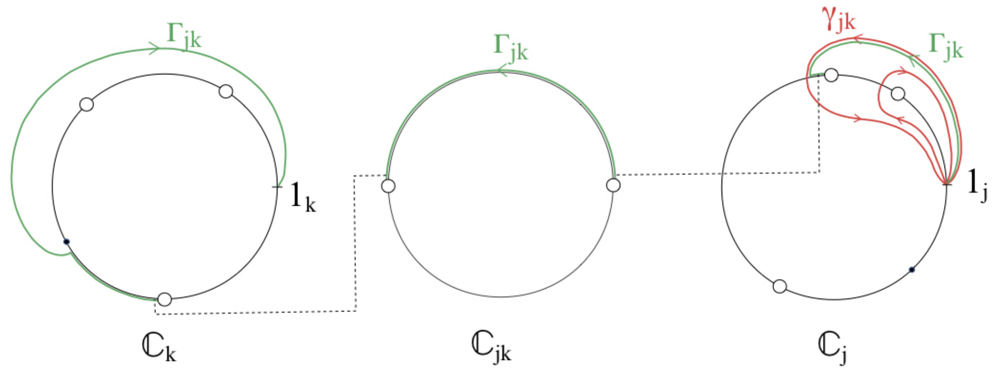

FIGURE 2. The paths γ_{jk} (red) and Γ_{jk} (green), for $(j,k)\in E^+$. The large circles represent the unit circles in $\overline{\mathbb{C}}_j$, $\overline{\mathbb{C}}_{jk}$ and $\overline{\mathbb{C}}_k$. The tiny circles represent disks that are removed when opening nodes. The bullets represent ends. The dots connect points that are identified when opening nodes.

(4) The circular arc from $z'_{kj}=-\varepsilon'$ to $z'_{kj}=-t_{kj}/\varepsilon'$. Its endpoint was identified with $z_{kj}=-\varepsilon'$ when opening nodes.
(5) The circular arc from $z_{kj}=-\varepsilon'$ to $z=e^{i\varepsilon}u_{kj'}$, where j' is the predecessor of j for the order $\prec$ on $E_k\cup R_k$ (or to $z=1_k$ in case j is the minimum of $E_k\cup R_k$).

(We could of course group paths (1) and (2) into one single arc, but it is convenient for the proof of Proposition 13 to write it this way.) If $\tau_{jk}<0$, some signs in the definition of β_{jk} must be changed, the result being that path number (3) is now on the lower half unit circle. All these paths are on the unit circle so $\sigma(\beta_{jk})=\beta_{jk}$. We define the path Γ_{jk} on $\overline{\Sigma}_{t,\mathrm{x}}$ from 1_j to 1_k as $\Gamma_{jk}=\alpha_{jk}\beta_{jk}\alpha_{kj'}^{-1}$ (or $\Gamma_{jk}=\alpha_{jk}\beta_{jk}$ in case j is the minimum of $E_k\cup R_k$). It transform as follows under σ:

$$\sigma(\Gamma_{jk})=\delta_{jk}^{-1}\,\Gamma_{jk}\,\delta'_{kj}. \tag{37}$$

8.2. **Formulation of the Monodromy Problem.** Let $\Sigma_{t,\mathrm{x}}$ be the Riemann surface $\overline{\Sigma}_{t,\mathrm{x}}$ minus the poles of $\xi_{t,\mathrm{x}}$, namely the ends p_{jk} for $(j,k)\in R$, the points $0_j,\infty_j$ for $j\in J$ and the points $q_{jk},\sigma(q_{jk})$ for $(j,k)\in E^+$. Fix an arbitrary $j_0\in J$ and take $z_0=1_{j_0}$ as base point.

Proposition 9. *Assume that the Regularity Problem is solved and that*

$$\forall(j,k)\in E\cup R,\quad \begin{cases}\mathcal{P}(\xi_{t,\mathrm{x}},\gamma_{jk})\in\Lambda SU(2)\\ \mathcal{P}(\xi_{t,\mathrm{x}},\gamma_{jk})\mid_{\lambda=1}=I_2\\ \frac{\partial}{\partial\lambda}\mathcal{P}(\xi_{t,\mathrm{x}},\gamma_{jk})\mid_{\lambda=1}=0\end{cases} \tag{38}$$

$$\forall(j,k)\in E^+,\quad \begin{cases}\mathcal{P}(\xi_{t,\mathrm{x}},\Gamma_{jk})\in\Lambda SU(2)\\ \mathcal{P}(\xi_{t,\mathrm{x}},\Gamma_{jk})\mid_{\lambda=1}=\pm I_2\\ \mathcal{P}(\xi_{t,\mathrm{x}},\Gamma_{jk})^{-1}\frac{\partial}{\partial\lambda}\mathcal{P}(\xi_{t,\mathrm{x}},\Gamma_{jk})\mid_{\lambda=1}=i(V_k-V_j)\end{cases} \tag{39}$$

where V_j for $j\in J$ are arbitrary matrices in $\mathfrak{su}(2)$. Then the Monodromy Problem (8) is solved.

Proof: for $j\in J$, let γ_{0_j} be a closed loop around 0_j in the unit disk of $\overline{\mathbb{C}}_j$, with base point 1_j. For $(j,k)\in E^+$, let $\gamma_{q_{jk}}$ be a closed loop with base point 1_j defined as follows: Items (1) and (2) in the definition of β_{jk} from 1_j to $z'_{jk}=\varepsilon$ in $\overline{\mathbb{C}}_{jk}$, then a closed loop in the unit disk of $\overline{\mathbb{C}}_{jk}$ around q_{jk}, and back to 1_j by the same path. Provided the Regularity Problem at 0_j and q_{jk} are solved, the gauged potentials

$\xi_{t,\mathrm{x}} \cdot G_j$ and $\xi_{t,\mathrm{x}} \cdot G_{jk}$ have trivial monodromy around 0_j and q_{jk}, respectively. Because the gauges have multivaluation $-I_2$ around these points, we have

$$\mathcal{P}(\xi_{t,\mathrm{x}}, \gamma_{0_j}) = \mathcal{P}(\xi_{t,\mathrm{x}}, \gamma_{q_{jk}}) = -I_2.$$

Any element of $\pi_1(\Sigma_{t,\mathrm{x}}, z_0)$ can be written as a product of the following paths or their inverse:

(1) γ_{jk} for $(j,k) \in E \cup R$,
(2) γ_{0_j} for $j \in J$,
(3) $\gamma_{q_{jk}}$ for $(j,k) \in E^+$,
(4) Γ_{jk} for $(j,k) \in E^+$.

Let $c \in \pi_1(\Sigma_{t,\mathrm{x}}, z_0)$ and decompose it as

$$c = \prod_{i=0}^{n-1} c_i$$

where each c_i or c_i^{-1} is a path in the above list. Then

$$\mathcal{P}(\xi_{t,\mathrm{x}}, c) = \prod_{i=0}^{n-1} \mathcal{P}(\xi_{t,\mathrm{x}}, c_i)$$

so we immediately see that the first two items of the Monodromy Problem (7) are solved. Each path c_i goes from a point 1_{j_i} to a point $1_{j_{i+1}}$, with $j_{i+1} = j_i$ for paths of type (1), (2) and (3) and $j_n = j_0$. Then we always have

$$\mathcal{P}(\xi_{t,\mathrm{x}}, c_i)^{-1} \frac{\partial}{\partial \lambda} \mathcal{P}(\xi_{t,\mathrm{x}}, c_i) \mid_{\lambda=1} = i(V_{j_{i+1}} - V_{j_i}).$$

Indeed, boths sides are zero for paths of type (1), (2), (3), and for paths of type (4) this follows from Equation (39). Consequently (using that $\pm I_2$ commutes with everything)

$$\mathcal{P}(\xi_{t,\mathrm{x}}, c)^{-1} \frac{\partial}{\partial \lambda} \mathcal{P}(\xi_{t,\mathrm{x}}, c) \mid_{\lambda=1} = \sum_{i=0}^{n-1} \mathcal{P}(\xi_{t,\mathrm{x}}, c_i)^{-1} \frac{\partial}{\partial \lambda} \mathcal{P}(\xi_{t,\mathrm{x}}, c_i) \mid_{\lambda=1} = i(V_{j_n} - V_{j_0}) = 0. \tag{40}$$

□

We shall take the following choice for the matrices V_j:

$$V_j = \frac{-i}{2} \begin{pmatrix} \mathrm{Re}(v_j) & -i\,\mathrm{Im}(v_j) \\ i\,\mathrm{Im}(v_j) & -\mathrm{Re}(v_j) \end{pmatrix} \tag{41}$$

where v_j denotes the vertices of the given graph Γ. Then for $(j,k) \in E^+$, we have $v_k - v_j = \ell_{jk} u_{jk}$ so

$$V_k - V_j = \frac{-i\,\ell_{jk}}{2} \begin{pmatrix} \mathrm{Re}(u_{jk}) & -i\,\mathrm{Im}(u_{jk}) \\ i\,\mathrm{Im}(u_j k) & -Re(u_{jk}) \end{pmatrix} = -\frac{\ell_{jk}}{2} N^S(u_{jk}). \tag{42}$$

Remark 7. (1) There is geometry behind our choice for V_j: we are in fact requiring that the image of 1_j by the immersion is v_j for all $j \in J$, up to a rigid motion: see Point (2) of Proposition 18.
(2) If the Regularity Problem at 0_j and ∞_j is solved, then Equations (38) for $k \in E_j \cup R_j$ are not independent, as the fundamental group of the n-punctured sphere has $n-1$ generators. We will still solve Problems (38) for all $k \in E_j \cup R_j$ and infer in Point (3) of Proposition 18 that the Regularity Problem at 0_j and ∞_j is solved. A similar remark holds for the Regularity Problem at q_{jk}.

8.3. **The renormalized γ-Monodromy.** In this section, we address the Monodromy Problem (38) for the curves γ_{jk}, $(j,k) \in E \cup R$. To compensate for the lack of symmetry of γ_{jk} (see Equation (36)), we conjugate $\mathcal{P}(\xi_{t,\mathrm{x}}, \gamma_{jk})$ by $\mathcal{P}(\xi_{t,\mathrm{x}}, \delta'_{jk})^{1/2}$ and define

$$\widetilde{M}_{jk}(t,\mathrm{x}) = \mathcal{P}(\xi_{t,\mathrm{x}}, \delta'_{jk})^{1/2}\mathcal{P}(\xi_{t,\mathrm{x}}, \gamma_{jk})\mathcal{P}(\xi_{t,\mathrm{x}}, \delta'_{jk})^{-1/2}.$$

Note that the square root is well-defined for t small enough because at $t=0$, $\xi_{0,\mathrm{x}}$ is holomorphic at p_{jk} for all $(j,k) \in E \cup R$ so $\mathcal{P}(\xi_{0,\mathrm{x}}, \delta'_{jk}) = I_2$. As in [28], we define for $t \neq 0$:

$$\widehat{M}_{jk}(t,\mathrm{x}) = t^{-1} \log \widetilde{M}_{jk}(t,x).$$

Proposition 10. *For $(j,k) \in E \cup R$:*

(1) *The renormalized monodromy $\widehat{M}_{jk}(t,\mathrm{x})$ extends at $t=0$ to an analytic map of (t,x) in a neighborhood of $(0,\underline{\mathrm{x}})$ with value in $\Lambda sl(2,\mathbb{C})$.*

(2) *$\widehat{M}_{jk}(t,\mathrm{x})$ has the symmetry*

$$\overline{\widehat{M}_{jk}} = -D\widehat{M}_{jk}D^{-1}. \tag{43}$$

(3) *Problem (38) is equivalent to the following problem for $(j,k) \in E \cup R$:*

$$\begin{cases} \widehat{M}_{jk}(t,\mathrm{x}) \in \Lambda\mathfrak{su}(2) & (i) \\ \widehat{M}_{jk}(t,\mathrm{x}) \mid_{\lambda=1} = 0 & (ii) \\ \frac{\partial}{\partial\lambda}\widehat{M}_{jk}(t,\mathrm{x}) \mid_{\lambda=1} = 0 & (iii) \end{cases} \tag{44}$$

(4) *At $t=0$, we have*

$$\widehat{M}_{jk}(0,\mathrm{x}) = 2\pi i \operatorname{Res}_{p_{jk}}\left[\Phi^S \frac{\partial\xi_{t,\mathrm{x}}}{\partial t} \mid_{t=0} (\Phi^S)^{-1}\right]. \tag{45}$$

Proof:

(1) By standard ODE theory, $\widetilde{M}_{jk}$ is an analytic map of all parameters. At $t=0$, $\widetilde{M}_{jk}(0,\mathrm{x}) = I_2$, so $\widehat{M}_{jk}$ extends analytically at $t=0$.

(2) By Proposition 2 and Equations (35), (36), we have

$$\overline{\mathcal{P}(\xi_{t,\mathrm{x}}, \delta_{jk})} = D\mathcal{P}(\xi_{t,\mathrm{x}}, \delta_{jk})^{-1}D^{-1} \tag{46}$$

$$\overline{\mathcal{P}(\xi_{t,\mathrm{x}}, \gamma_{jk})} = D\mathcal{P}(\xi_{t,\mathrm{x}}, \delta'_{jk})\mathcal{P}(\xi_{t,\mathrm{x}}, \gamma_{jk})^{-1}\mathcal{P}(\xi_{t,\mathrm{x}}, \delta'_{jk})^{-1}D^{-1}. \tag{47}$$

Hence $\widetilde{M}_{jk}$ has the symmetry

$$\overline{\widetilde{M}_{jk}(t,\mathrm{x})} = D\widetilde{M}_{jk}(t,\mathrm{x})^{-1}D^{-1}.$$

Point (2) follows by taking the logarithm, remembering that $t \in \mathbb{R}$.

(3) Assuming that $\mathcal{P}(\xi_{t,\mathrm{x}}, \delta'_{jk})$ solves the Monodromy Problem (8), the Monodromy Problem for $\mathcal{P}(\xi_{t,\mathrm{x}}, \gamma_{jk})$ is equivalent to

$$\begin{cases} \widetilde{M}_{jk}(t,\mathrm{x}) \in \Lambda SU(2) \\ \widetilde{M}_{jk}(t,\mathrm{x}) \mid_{\lambda=1} = I_2 \\ \frac{\partial}{\partial\lambda}\widetilde{M}_{jk}(t,\mathrm{x}) \mid_{\lambda=1} = 0 \end{cases}$$

which, taking the logarithm, is equivalent to Problem (44). Remembering the definition of δ'_{jk}, Point (3) follows by induction on k for the order $\prec$ on $E_j \cup R_j$.

(4) We have, since $\mathcal{P}(\xi_{0,\mathrm{x}},\gamma_{jk})=\mathcal{P}(\xi_{0,\mathrm{x}},\delta'_{jk})=I_2$:

$$\widehat{M}_{jk}(0,\mathrm{x})=\frac{\partial}{\partial t}\widetilde{M}_{jk}(t,\mathrm{x})\mid_{t=0}=\frac{\partial}{\partial t}\mathcal{P}(\xi_{t,\mathrm{x}},\gamma_{jk})\mid_{t=0}.$$

At $t=0$, we have $A_j=0$ and $B_j=C_j=1/2$ so $\xi_{0,\mathrm{x}}=\xi^S$. By Proposition 8 in [28], we obtain

$$\frac{\partial}{\partial t}\mathcal{P}(\xi_{t,\mathrm{x}},\gamma_{jk})\mid_{t=0}=\int_{\gamma_{jk}}\Phi^S\frac{\partial\xi_{t,\mathrm{x}}}{\partial t}\mid_{t=0}(\Phi^S)^{-1}=2\pi i\operatorname{Res}_{p_{jk}}\Phi^S\frac{\partial\xi_{t,\mathrm{x}}}{\partial t}\mid_{t=0}(\Phi^S)^{-1}.$$

□

8.4. **The Monodromy Problem around nodes.** In this section we fix $(j,k)\in E$ and solve Problem (44). Let $U_{jk}=\Phi^S(p_{jk})$. In view of Equation (45), it is advantageous to conjugate $\widehat{M}_{jk}$ by the inverse of U_{jk}. Since $p_{jk}\in\mathbb{S}^1$, $U_{jk}\in\Lambda SU(2)$ and $\overline{U_{jk}}=DU_{jk}D^{-1}$ by Equation (13). So this conjugation does not affect the Monodromy Problem (44) nor the symmetry (43). We define

$$\breve{M}_{jk}(t,\mathrm{x})=U_{jk}^{-1}\widehat{M}_{jk}(t,\mathrm{x})U_{jk}$$

$$\mathcal{F}_{jk}(t,\mathrm{x})=i\left(\breve{M}_{jk;11}(t,\mathrm{x})+\breve{M}_{jk;11}(t,\mathrm{x})^*\right)\tag{48}$$

$$\mathcal{G}_{jk}(t,\mathrm{x})=\lambda\left(\breve{M}_{jk;12}(t,\mathrm{x})+\breve{M}_{jk;21}(t,\mathrm{x})^*\right).\tag{49}$$

so that $\breve{M}_{jk}\in\Lambda\mathfrak{su}(2)$ is equivalent to $\mathcal{F}_{jk}=\mathcal{G}_{jk}=0$. By symmetry (43), $\mathcal{F}_{jk}(t,\mathrm{x})$ and $\mathcal{G}_{jk}(t,\mathrm{x})$ are in $\mathcal{W}_{\mathbb{R}}$. By definition, $\mathcal{F}_{jk}^*=-\mathcal{F}_{jk}$ so since $\mathcal{F}_{jk}^0\in\mathbb{R}$, we have $\mathcal{F}_{jk}^0=0$ and we do not need to solve $\mathcal{F}_{jk}^-=0$. The σ-symmetry gives us one more piece of information: if $\breve{M}_{jk}\in\Lambda\mathfrak{su}(2)$, then the symmetry (43) and the definition of the conjugation and star operators give

$$\breve{M}_{jk;11}(\lambda)=-\breve{M}^*_{jk;11}(\lambda)=\overline{\breve{M}^*_{jk;11}}(\lambda)=\breve{M}_{jk;11}(1/\lambda).$$

This implies

$$\frac{\partial}{\partial\lambda}\breve{M}_{jk;11}\mid_{\lambda=1}=0.$$

We define

$$\mathcal{E}_{1,jk}=(\mathcal{E}_{1,jk,i})_{1\le i\le 6}=\left[\mathcal{F}^+_{jk},\mathcal{G}^+_{jk},\lambda(\mathcal{G}^{\le 0}_{jk})^*,i\breve{M}_{jk;11}\mid_{\lambda=1},\breve{M}_{jk;21}\mid_{\lambda=1},\frac{\partial}{\partial\lambda}\breve{M}_{jk;21}\mid_{\lambda=1}\right]\tag{50}$$

$$\mathrm{x}_{1,jk}=\left(a^+_{jk},b^+_{jk},c^+_{jk},a^0_{jk},b^0_{jk},c^0_{jk}\right)$$

so Problem (44) is equivalent to $\mathcal{E}_{1,jk}(t,\mathrm{x})=0$.

Proposition 11. *For $(j,k)\in E$:*

1. $\mathcal{E}_{1,jk}(t,\mathrm{x})\in(\mathcal{W}_{\mathbb{R}}^{>0})^3\times\mathbb{R}^3$.
2. *At the central value,* $\mathcal{E}_{1,jk}(0,\underline{\mathrm{x}})=0$.
3. *The partial differential of $\mathcal{E}_{1,jk}$ with respect to $\mathrm{x}_{1,jk}$ at $(0,\underline{\mathrm{x}})$ is an automorphism of $(\mathcal{W}_{\mathbb{R}}^{>0})^3\times\mathbb{R}^3$.*
4. *The full differential of $\mathcal{E}_{1,jk}$ with respect to x at $(0,\underline{\mathrm{x}})$ only involves the variables $\mathrm{x}_{1,jk}$, r_{jk}, A_{jk} and C_{jk}.*
5. *If $X\in\operatorname{Ker}(d_{\mathrm{x}}\mathcal{E}_{1,jk}(0,\underline{\mathrm{x}}))$ satisfies $dA_{jk}(X)=0$, then $db_{jk}(X)=dc_{jk}(X)=0$.*

Proof: Point (1) follows from symmetry. By Propositions 3 and 4, we have in a neighborhood of p_{jk}:

$$\frac{\partial}{\partial t}\xi_{t,\mathrm{x}}\mid_{t=0}=m_{jk}\frac{dz}{z-p_{jk}}+r_{jk}M_{jk}\frac{1+q_{jk}^2}{1-q_{jk}^2}\frac{p_{jk}dz}{(z-p_{jk})^2}+\text{ holomorphic.}$$

A simple computation gives

$$\frac{\partial}{\partial z}\left(\Phi^S(z)M_{jk}\Phi^S(z)^{-1}\right)|_{z=p_{jk}}= p_{jk}^{-1}\Phi^S(p_{jk})[M_j,M_{jk}]\Phi^S(p_{jk})^{-1}.$$

(Here M_j and M_{jk} have their values at $t=0$.) Equation (45) gives

$$\breve{M}_{jk}(0,\mathrm{x}) = 2\pi i\, m_{jk} + 2\pi i\, r_{jk}\frac{1+q_{jk}^2}{1-q_{jk}^2}\left[M_j, M_{jk}\right].$$

Observe that the partial differential of $\breve{M}_{jk}$ with respect to q_{jk} is zero since $q_{jk}=0$ at the central value. Point (4) follows. Assume from now on that $q_{jk}=0$ and $A_{jk}=0$. By Equation (31), $B_{jk}=\frac{1}{4C_{jk}}$. We obtain

(51) $$\breve{M}_{jk}(0,\mathrm{x}) = 2\pi i\begin{pmatrix} a_{jk} & \lambda^{-1}ib_{jk}\\ ic_{jk} & -a_{jk}\end{pmatrix} + \pi i\, r_{jk}\begin{pmatrix}\lambda^{-1}C_{jk}-\frac{\lambda}{4C_{jk}} & 0\\ 0 & \frac{\lambda}{4C_{jk}}-\lambda^{-1}C_{jk}\end{pmatrix}.$$

In particular at the central value, this simplifies to

(52) $$\breve{M}_{jk}(0,\underline{\mathrm{x}}) = 2\pi i\,\tau_{jk}\frac{(\lambda-1)^2}{4\lambda}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\in\Lambda\mathfrak{su}(2).$$

which proves Point (2). To prove Point (3), assume that $r_{jk}=\tau_{jk}$ and $C_{jk}=1/2$ are fixed. Differentiating Equation (51) at $(t,\mathrm{x})=(0,\underline{\mathrm{x}})$ we obtain:

$$d\mathcal{F}_{jk} = -2\pi\left(da_{jk}-da_{jk}^*\right)$$

$$d\mathcal{G}_{jk} = -2\pi(db_{jk}+\lambda dc_{jk}^*)$$

$$d\mathcal{E}_{1,jk,1} = -2\pi\, da_{jk}^+$$

$$d\mathcal{E}_{1,jk,2} = -2\pi(db_{jk}^+ + \lambda dc_{jk}^0)$$

$$d\mathcal{E}_{1,jk,3} = -2\pi(dc_{jk}^+ + \lambda db_{jk}^0)$$

$$d\mathcal{E}_{1,jk,4} - d\mathcal{E}_{1,jk,1}\,|_{\lambda=1} = -2\pi\, da_{jk}^0$$

$$d\mathcal{E}_{1,jk,5} - d\mathcal{E}_{1,jk,3}\,|_{\lambda=1} = -2\pi(dc_{jk}^0 - db_{jk}^0)$$

$$d\mathcal{E}_{1,jk,6} - \frac{\partial}{\partial\lambda}d\mathcal{E}_{1,jk,3}\,|_{\lambda=1} = 2\pi db_{jk}^0.$$

Point (3) easily follows from these formulas. Finally, to prove Point (5), relax the hypothesis $r_{jk}=\tau_{jk}$ and $C_{jk}=1/2$. By Equation (51), the off-diagonal part of $\breve{M}_{jk}$ does not change, so $d\mathcal{E}_{1,jk,i}$ for $i\in\{2,3,5,6\}$ do not change. Since these equations determine b_{jk} and c_{jk}, we obtain $db_{jk}(X)=dc_{jk}(X)=0$. □

Remark 8. We will solve all equations at the same time by one single application of the Implicit Function Theorem in Section 9.

8.5. **The Monodromy Problem around ends.** In this section we fix $(j,k)\in R$ and solve Problem (44). We follow closely the resolution of the same problem in [28]. We cannot take $U_{jk}=\Phi^S(p_{jk})$ because $p_{jk}\not\in\mathbb{S}^1$ so we take $U_{jk}=\Phi^S(u_{jk})$ and conjugate $\widehat{M}_{jk}$ by the inverse of U_{jk}. Observe that if $a_{jk}=b_{jk}=0$ then $\xi_{t,\mathrm{x}}$ is holomorphic at p_{jk} so $\widehat{M}_{jk}=0$. This prompts us to take

$$a_{jk}=(\lambda-1)^2\,\widehat{a}_{jk}\quad\text{ and }\quad b_{jk}=(\lambda-1)^2\,\widehat{b}_{jk}$$

with $\widehat{a}_{jk},\widehat{b}_{jk}\in\mathcal{W}_{\mathbb{R}}^{\geq 0}$. This way, Points (ii) and (iii) of Problem (44) are automatically satisfied. We define

$$\widetilde{M}_{jk}(t,\mathrm{x})=\frac{\lambda}{(\lambda-1)^2}U_{jk}^{-1}\widehat{M}_{jk}(t,\mathrm{x})U_{jk}$$

which extends at $\lambda=1$ to an analytic map of (t,x) (see details in Section 6.2 of [28]). Since $(\lambda-1)^2/\lambda$ is unitary on the unit circle, Point (i) of Problem (44) is equivalent to $\widetilde{M}_{jk}(t,\mathrm{x})\in\Lambda\mathfrak{su}(2)$. Define $\mathcal{F}_{jk}$ and $\mathcal{G}_{jk}$ by Equations (48) and (49) and

$$\mathcal{E}_{2,jk}=(\mathcal{E}_{2,jk,i})_{1\leq i\leq 4}=\left[\mathcal{F}_{jk}^{+},\mathcal{G}_{jk}^{+},(\mathcal{G}_{jk}^{-})^{*},\mathcal{G}_{jk}^{0}\right].$$

Problem (44) is equivalent to $\mathcal{E}_{2,jk}(t,\mathrm{x})=0$. Writing $p_{jk}=e^{i\theta_{jk}}$ with $\theta_{jk}\in\mathcal{W}_{\mathbb{R}}^{\geq 0}$, we define

$$\mathrm{x}_{2,jk}=(\widehat{a}_{jk}^{+},\widehat{b}_{jk}^{+},\theta_{jk}^{+},\widehat{b}_{jk}^{0}).$$

Proposition 12. *For $(j,k)\in R$:*
(1) $\mathcal{E}_{2,jk}(t,\mathrm{x})\in(\mathcal{W}_{\mathbb{R}}^{>0})^3\times\mathbb{R}$.
(2) *At the central value,* $\mathcal{E}_{2,jk}(0,\underline{\mathrm{x}})=0$.
(3) *The differential of $\mathcal{E}_{2,jk}$ with respect to x at $(0,\underline{\mathrm{x}})$ only involves the variable $\mathrm{x}_{2,jk}$ and is an automorphism of $(\mathcal{W}_{\mathbb{R}}^{>0})^3\times\mathbb{R}$.*

Proof: Point (1) follows from symmetry. Equation (45) gives

$$\widetilde{M}_{jk}(0,\mathrm{x})=2\pi i\lambda\,U_{jk}^{-1}\mathrm{Res}_{p_{jk}}\left[\Phi^S(z)\begin{pmatrix}0&0\\1&0\end{pmatrix}\Phi^S(z)^{-1}\left(\frac{\widehat{a}_{jk}p_{jk}}{(z-p_{jk})^2}+\frac{i\widehat{b}_{jk}}{z-p_{jk}}\right)\right]U_{jk}.$$

A simple computation gives

$$\frac{\partial}{\partial z}\Phi^S(z)\begin{pmatrix}0&0\\1&0\end{pmatrix}\Phi^S(z)^{-1}\mid_{z=p_{jk}}=\frac{\lambda^{-1}}{2p_{jk}}\Phi^S(p_{jk})\begin{pmatrix}1&0\\0&-1\end{pmatrix}\Phi^S(p_{jk})^{-1}.$$

This gives

$$\widetilde{M}_{jk}(0,\mathrm{x})=2\pi i\,U_{jk}^{-1}\Phi^S(p_{jk})\begin{pmatrix}\widehat{a}_{jk}/2&0\\\lambda i\widehat{b}_{jk}&-\widehat{a}_{jk}/2\end{pmatrix}\Phi^S(p_{jk})^{-1}U_{jk}.\tag{53}$$

At the central value, this simplifies to

$$\widetilde{M}_{jk}(0,\underline{\mathrm{x}})=2\pi i\begin{pmatrix}\tau_{jk}/4&0\\0&-\tau_{jk}/4\end{pmatrix}\in\Lambda\mathfrak{su}(2)$$

which proves Point (2). Using Equation (53), we obtain at the central value

$$\frac{\partial}{\partial p_{jk}}\widetilde{M}_{jk}=2\pi i\,u_{jk}^{-1}\left[\begin{pmatrix}0&\lambda^{-1}/2\\\lambda/2&0\end{pmatrix},\begin{pmatrix}\tau_{jk}/4&0\\0&-\tau_{jk}/4\end{pmatrix}\right]=\frac{2\pi i\,\tau_{jk}}{4u_{jk}}\begin{pmatrix}0&-\lambda^{-1}\\\lambda&0\end{pmatrix}.$$

Hence by the chain rule, since $dp_{jk}/d\theta_{jk}=iu_{jk}$ at $\mathrm{x}=\underline{\mathrm{x}}$:

$$d_{\mathrm{x}}\widetilde{M}_{jk}(0,\underline{\mathrm{x}})=2\pi i\begin{pmatrix}d\widehat{a}_{jk}/2&-\lambda^{-1}i\tau_{jk}\,d\theta_{jk}/4\\\lambda i\,d\widehat{b}_{jk}+\lambda i\tau_{jk}\,d\theta_{jk}/4&-d\widehat{a}_{jk}/2\end{pmatrix}$$

which gives

$$d\mathcal{E}_{2,jk,1} = -\pi\, d\widehat{a}^+_{jk}$$
$$d\mathcal{E}_{2,jk,2} = \frac{\pi}{2}\tau_{jk}\, d\theta^+_{jk}$$
$$d\mathcal{E}_{2,jk,3} = -2\pi d\widehat{b}^+_{jk} - \frac{\pi}{2}\tau_{jk}\, d\theta^+_{jk}$$
$$d\mathcal{E}_{2,jk,4} = -2\pi\, d\widehat{b}^0_{jk}.$$

Point (3) easily follows. □

8.6. **The Γ-Monodromy Problem.** In this section, we fix $(j,k) \in E^+$ and we solve Problem (39). To compensate for the lack of symmetry of Γ_{jk} (see Equation (37)), we multiply $\mathcal{P}(\xi_{t,\mathrm{x}}, \Gamma_{jk})$ by suitable (different) factors on the left and right. Then we conjugate by $\Phi^S(u_{jk}) \in \Lambda SU(2)$ to simplify computations. We define for $t > 0$:

$$P_{jk}(t,\mathrm{x}) = \Phi^S(u_{jk})^{-1}\mathcal{P}(\xi_{t,\mathrm{x}}, \delta_{jk})^{-1/2}\mathcal{P}(\xi_{t,\mathrm{x}}, \Gamma_{jk})\mathcal{P}(\xi_{t,\mathrm{x}}, \delta'_{kj})^{1/2}\Phi^S(u_{jk}).$$

Definition 11. Let $f(t)$ be a function of the real variable $t \geq 0$. We say that f is a smooth function of t and $t\log t$ if there exists a smooth function of two variables $g(t,s)$ defined in a neighborhood of $(0,0)$ in $\mathbb{R}^2$ such that $f(t) = g(t, t\log t)$ for $t > 0$ and $f(0) = g(0,0)$.

Remark 9. The function $t\log t$ extends continuously at 0 but the extension is not differentiable at 0 and is only of Hölder class $C^{0,\alpha}$ for all $\alpha \in (0,1)$. Therefore, a smooth function of t and $t\log t$ is only of class $C^{0,\alpha}$ and is not differentiable at $t = 0$.

Proposition 13. (1) *$P_{jk}(t,\mathrm{x})$ has the symmetry*

$$\overline{P_{jk}} = DP_{jk}D^{-1}.$$

(2) *$P_{jk}(t,\mathrm{x})$ extends at $t = 0$ to a smooth function of t, $t\log t$ and* x. *Moreover, we have at $t = 0$:*

$$P_{jk}(0,\mathrm{x}) = \Phi^S(u_{jk})^{-1}\Phi^S(p_{jk})\exp\left(M_{jk}\int_1^{-1}\omega_{q_{jk}}\right)\Phi^S(p_{kj})^{-1}\Phi^S(u_{jk}).$$

(3) *At the central value*

$$P_{jk}(0,\underline{\mathrm{x}}) = \pm\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \in \Lambda SU(2).$$

(4) *Provided Problem* (38) *is solved, Problem* (39) *is equivalent to*

$$(54)\qquad \begin{cases} P_{jk}(t,\mathrm{x}) \in \Lambda SU(2) & (i)\\ P_{jk}(t,\mathrm{x})\mid_{\lambda=1} = \pm I_2 & (ii)\\ P_{jk}(t,\mathrm{x})^{-1}\frac{\partial}{\partial\lambda}P_{jk}(t,\mathrm{x})\mid_{\lambda=1} = \dfrac{\ell_{jk}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} & (iii)\end{cases}$$

Proof:

(1) Equation (37) and Proposition 2 give

$$\overline{\mathcal{P}(\xi_{t,\mathrm{x}}, \Gamma_{jk})} = D\mathcal{P}(\xi_{t,\mathrm{x}}, \delta_{jk})^{-1}\mathcal{P}(\xi_{t,\mathrm{x}}, \Gamma_{jk})\mathcal{P}(\xi_{t,\mathrm{x}}, \delta'_{kj})D^{-1}.$$

Using the symmetry (46) and $\overline{\Phi^S(u_{jk})} = D\Phi^S(u_{jk})D^{-1}$, we obtain Point (1).

(2) The function $\mathcal{P}(\xi_{t,\mathrm{x}}, \alpha_{jk})$ is an analytic function of all parameters by Theorem 3 because the path α_{jk} stays away from the nodes. The same holds for the paths number (1), (3), (5) in the definition of the path β_{jk} and the path $\alpha_{kj'}$. By Theorem 5 in Appendix B (see also Remark 14), the principal solution of $\xi_{t,\mathrm{x}}$ on path number (2) extends at $t = 0$ to a smooth function of t, $t\log t$ and x, with the following value at $t = 0$:

$$\mathcal{P}(M_j\omega_0, z_{jk} = \varepsilon', z_{jk} = 0)\mathcal{P}(M_{jk}\omega_{q_{jk}}, z'_{jk} = 0, z'_{jk} = \varepsilon').$$

In the same way, the principal solution on path number (4) extends to a smooth function of t, $t\log t$ and x with the following value at $t=0$:

$$\mathcal{P}(M_{jk}\omega_{q_{jk}},z'_{kj}=-\varepsilon',z'_{kj}=0)\mathcal{P}(M_k\omega_0,z_{kj}=0,z_{kj}=-\varepsilon').$$

Collecting all terms, the function $\mathcal{P}(\xi_{t,\mathrm{x}},\Gamma_{jk})$ extends to a smooth function of t and $t\log t$ with the following value at $t=0$:

$$\begin{aligned} & \mathcal{P}(M_j\omega_0,1_j,e^{i\varepsilon}u_{jk})\mathcal{P}(M_j\omega_0,e^{i\varepsilon}u_{jk},z_{jk}=\varepsilon')\mathcal{P}(M_j\omega_0,z_{jk}=\varepsilon',z_{jk}=0)\\ & \times\mathcal{P}(M_{jk}\omega_{q_{jk}},z'_{jk}=0,z'_{jk}=\varepsilon')\mathcal{P}(M_{jk}\omega_{q_{jk}},z'_{jk}=\varepsilon',z'_{kj}=-\varepsilon')\mathcal{P}(M_{jk}\omega_{q_{jk}},z'_{kj}=-\varepsilon',z'_{kj}=0)\\ & \times\mathcal{P}(M_k\omega_0,z_{kj}=0,z_{kj}=-\varepsilon')\mathcal{P}(M_k\omega_0,z_{kj}=-\varepsilon',e^{i\varepsilon}u_{kj'})\mathcal{P}(M_k\omega_0,e^{i\varepsilon}u_{kj'},1_k)\\ = \; & \mathcal{P}(M_j\omega_0,1_j,p_{jk})\mathcal{P}(M_{jk}\omega_{q_{jk}},1,-1)\mathcal{P}(M_k\omega_0,p_{kj},1_k)\end{aligned}$$

In the above computation, M_j and M_{jk} have their value at $t=0$, so $M_j\omega_0=\xi^S$. Point (2) follows.

(3) At $(t,\mathrm{x})=(0,\underline{\mathrm{x}})$, we have $M_{jk}\omega_{q_{jk}}=\xi^C$ and $p_{jk}=-p_{kj}=u_{jk}$ so

$$P_{jk}(0,\underline{\mathrm{x}})=\Phi^C(-1)\Phi^S(-1)^{-1}=\pm\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}.$$

Remark 10. Note that Φ^S and Φ^C are both multivalued with multivaluation $\pm I_2$. This is why we put a $\pm$ sign in Point (3). We do not need to resolve this multivaluation.

(4) Assuming that Problem (38) is solved, Items (i) and (ii) of Problems (39) and (54) are clearly equivalent. Assuming Item (ii) is true, we have:

$$\frac{\partial}{\partial\lambda}P_{jk}(t,\mathrm{x})\mid_{\lambda=1}=\Phi^S(u_{jk})^{-1}\frac{\partial}{\partial\lambda}\mathcal{P}(\xi_{t,\mathrm{x}},\Gamma_{jk})\Phi^S(u_{jk})\mid_{\lambda=1}.$$

On the other hand, by Equations (5) and (42):

$$\frac{\ell_{jk}}{2}\Phi^S(u_{jk})\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi^S(u_{jk})^{-1}\mid_{\lambda=1}=\frac{-i\,\ell_{jk}}{2}N^S(u_{jk})=i(V_k-V_j)$$

So Items (iii) of Problems (39) and (54) are equivalent. □

We define for (t,x) in a neighborhoof of $(0,\underline{\mathrm{x}})$:

$$\widetilde{P}_{jk}(t,\mathrm{x})=\log(P_{jk}(t,\mathrm{x})P_{jk}(0,\underline{\mathrm{x}})^{-1}).$$

By Point (1) of Proposition 13, $\widetilde{P}_{jk}$ has the symmetry

$$\overline{\widetilde{P}_{jk}}=D\widetilde{P}_{jk}D^{-1}. \tag{55}$$

Proposition 14. *Problem* (54) *is equivalent to*

$$\begin{cases}\widetilde{P}_{jk}(t,\mathrm{x})\in\Lambda\mathfrak{su}(2) & (i)\\ \widetilde{P}_{jk;12}(t,\mathrm{x})\mid_{\lambda=1}=0 & (ii)\\ \frac{\partial}{\partial\lambda}\widetilde{P}_{jk}(t,\mathrm{x})\mid_{\lambda=1}=\dfrac{\ell_{jk}-2}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} & (iii)\end{cases} \tag{56}$$

Proof:

- Items (i) of Problems (54) and (56) are equivalent by Point (3) of Proposition 13.
- Item (ii) of Problem (54) is equivalent to $\widetilde{P}_{jk}\mid_{\lambda=1}=0$. Assuming that Item (i) of Problem (56) holds, we have by symmetry

$$\widetilde{P}_{jk;11}(1)=-\widetilde{P}^*_{jk;11}(1)=-\overline{\widetilde{P}_{jk;11}(1)}=-\widetilde{P}_{jk;11}(1)$$

so $\widetilde{P}_{jk;11}(1)=0$. Hence Items (ii) of Problems (54) and (56) are equivalent.

- Assuming Item (ii) of Problem (54) is satisfied, we have

$$P_{jk}(t,\mathrm{x}) = P_{jk}(0,\underline{\mathrm{x}}) = \pm I_2$$

so

$$\frac{\partial}{\partial\lambda}\widetilde{P}_{jk}(t,\mathrm{x})\mid_{\lambda=1} = P_{jk}(t,\mathrm{x})^{-1}\frac{\partial}{\partial\lambda}P_{jk}(t,\mathrm{x})\mid_{\lambda=1} - P_{jk}(0,\underline{\mathrm{x}})^{-1}\frac{\partial}{\partial\lambda}P_{jk}(0,\underline{\mathrm{x}})\mid_{\lambda=1}$$

and by Point (3) of Proposition 13,

$$P_{jk}(0,\underline{\mathrm{x}})^{-1}\frac{\partial}{\partial\lambda}P_{jk}(0,\underline{\mathrm{x}})\mid_{\lambda=1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So Items (iii) of Problem (54) and (56) are equivalent. □

We define

$$\mathcal{F}_{jk}(t,\mathrm{x}) = \widetilde{P}_{jk;11}(t,\mathrm{x}) + \widetilde{P}_{jk;11}(t,\mathrm{x})^*$$

$$\mathcal{G}_{jk}(t,\mathrm{x}) = i\left(\widetilde{P}_{jk;12}(t,\mathrm{x}) + \widetilde{P}_{jk;21}(t,\mathrm{x})^*\right).$$

By symmetry (55), $\mathcal{F}_{jk}(t,\mathrm{x})$ and $\mathcal{G}_{jk}(t,\mathrm{x})$ are in $\mathcal{W}_{\mathbb{R}}$. We define

$$\mathcal{E}_{3,jk} = (\mathcal{E}_{3,jk,i})\mid_{1\le i\le 7} = \left[\mathcal{F}^+_{jk}, \mathcal{G}^+_{jk}, (\mathcal{G}^-_{jk})^*, \mathcal{F}^0_{jk}, \mathcal{G}^0_{jk}, i\,\widetilde{P}_{jk;12}\mid_{\lambda=1}, i\,\frac{\partial\widetilde{P}_{jk;12}}{\partial\lambda}\mid_{\lambda=1}\right]$$

(57) $$\mathcal{L}_{jk}(t,\mathrm{x}) = \frac{\partial\widetilde{P}_{jk;11}(t,\mathrm{x})}{\partial\lambda}\mid_{\lambda=1} - \frac{(\ell_{jk}-2)}{2}.$$

Problem (56) is equivalent to $\mathcal{E}_{3,jk}(t,\mathrm{x}) = 0$ and $\mathcal{L}_{jk}(t,\mathrm{x}) = 0$. We leave aside the equation $\mathcal{L}_{jk}(t,\mathrm{x}) = 0$ for the moment and will solve it in Section 9 using the non-degeneracy hypothesis. Regarding the equation $\mathcal{E}_{3,jk} = 0$, recall that $q_{jk} \in i\mathcal{W}^{\ge 0}_{\mathbb{R}}$ and $p_{jk} = e^{i\theta_{jk}}$ with $\theta_{jk}\in\mathbb{R}$ and define

$$\mathrm{x}_{3,jk} = \left(A^+_{jk}, C^+_{jk}, \mathrm{Im}(q^+_{jk}), A^0_{jk}, C^0_{jk}, \theta_{jk}, \theta_{kj}\right).$$

Proposition 15. *For $(j,k)\in E^+$:*

1. $\mathcal{E}_{3,jk}(t,\mathrm{x}) \in (\mathcal{W}^{>0}_{\mathbb{R}})^3\times\mathbb{R}^4$.
2. *At the central value,* $\mathcal{E}_{3,jk}(0,\underline{\mathrm{x}}) = 0$.
3. *The partial differential of $\mathcal{E}_{3,jk}$ at $(0,\underline{\mathrm{x}})$ with respect to $\mathrm{x}_{3,jk}$ is an automorphism of $(\mathcal{W}^{>0}_{\mathbb{R}})^3\times\mathbb{R}^4$.*
4. *The full differential of $\mathcal{E}_{3,jk}$ at $(0,\underline{\mathrm{x}})$ only involves the variables $\mathrm{x}_{3,jk}$ and $\mathrm{Im}(q^0_{jk})$.*
5. *If $X\in\mathrm{Ker}(d\mathcal{E}_{3,jk}(0,\underline{\mathrm{x}}))$, then $dA_{jk}(X) = dC_{jk}(X) = 0$.*

Proof:

- Point (1) comes from symmetry.
- Point (2) is clear since $\widetilde{P}_{jk}(0,\underline{\mathrm{x}}) = 0$ by definition.
- We have at $t=0$

$$\widetilde{P}_{jk}(0,\mathrm{x}) = \log\left[\Phi^S(u_{jk})^{-1}\Phi^S(p_{jk})\exp\left(M_{jk}\int_1^{-1}\omega_{q_{jk}}\right)\Phi^S(p_{kj})^{-1}\Phi^S(u_{kj})\Phi^C(-1)^{-1}\right].$$

  Point (4) follows by inspection.
- We set $q_{jk}=0$ to compute the partial derivatives with respect to all parameters but q_{jk}. By Equation (31), at $t=0$ we have $M^2_{jk} = \frac{1}{4}I_2$ so

$$\exp\left(M_{jk}\int_1^{-1}\omega_0\right) = \exp(\pi i M_{jk}) = 2iM_{jk}.$$

By Equation (31) we have $\partial B_{jk}/\partial A_{jk} = 0$ and $\partial B_{jk}/\partial C_{jk} = -1$ at $\mathrm{x} = \underline{\mathrm{x}}$. This gives by the chain rule

$$\frac{\partial \widetilde{P}_{jk}}{\partial A_{jk}}(0,\underline{\mathrm{x}}) = 2i\frac{\partial M_{jk}}{\partial A_{jk}}\Phi^C(-1)^{-1} = 2i\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}$$

$$\frac{\partial \widetilde{P}_{jk}}{\partial C_{jk}}(0,\underline{\mathrm{x}}) = 2i\frac{\partial M_{jk}}{\partial C_{jk}}\Phi^C(-1)^{-1} = 2i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\frac{\partial \widetilde{P}_{jk}}{\partial \theta_{jk}}(0,\underline{\mathrm{x}}) = \xi^S(u_{jk})\frac{\partial p_{jk}}{\partial \theta_{jk}} = \begin{pmatrix} 0 & \lambda^{-1}i/2 \\ \lambda\, i/2 & 0 \end{pmatrix}$$

$$\frac{\partial \widetilde{P}_{jk}}{\partial \theta_{kj}}(0,\underline{\mathrm{x}}) = -\Phi^C(-1)\xi^S(u_{kj})\frac{\partial p_{kj}}{\partial \theta_{kj}}\Phi^C(-1)^{-1} = \begin{pmatrix} 0 & -\lambda\, i/2 \\ -\lambda^{-1}i/2 & 0 \end{pmatrix}.$$

- Next we compute the partial derivative with respect to q_{jk} at $(0,\underline{\mathrm{x}})$:

$$\frac{\partial}{\partial q_{jk}}\int_1^{-1}\omega_{q_{jk}} = \frac{\partial}{\partial q_{jk}}\int_1^{-1}\frac{dz}{z-q_{jk}} - \frac{q_{jk}\,dz}{1+q_{jk}z}\int_1^{-1}\frac{dz}{z^2} - dz = 4$$

$$\frac{\partial \widetilde{P}_{jk}}{\partial q_{jk}}(0,\underline{\mathrm{x}}) = 4\,\Phi^C(-1)\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}\Phi^C(-1)^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

- Write $q_{jk} = i\nu_{jk}$ with $\nu_{jk} \in \mathcal{W}_{\mathbb{R}}^{\geq 0}$. Remembering that $\theta_{jk}, \theta_{kj} \in \mathbb{R}$ we obtain at $(0,\underline{x})$:

$$d\mathcal{F}_{jk} = -2dC_{jk} - 2dC_{jk}^*$$

$$d\mathcal{G}_{jk} = -2(dA_{jk} + dA_{jk}^*) - 2(d\nu_{jk} - d\nu_{jk}^*)$$

$$d\mathcal{E}_{3,jk,1} = -2dC_{jk}^+$$

$$d\mathcal{E}_{3,jk,2} = -2dA_{jk}^+ - 2d\nu_{jk}^+$$

$$d\mathcal{E}_{3,jk,3} = -2dA_{jk}^+ + 2d\nu_{jk}^+$$

$$d\mathcal{E}_{3,jk,4} = -4dC_{jk}^0$$

$$d\mathcal{E}_{3,jk,5} = -4dA_{jk}^0.$$

If $X \in \mathrm{Ker}(d\mathcal{E}_{3,jk}(0,\underline{\mathrm{x}}))$, we obtain from these formulas $dA_{jk}(X) = dC_{jk}(X) = d\nu_{jk}^+(X) = 0$, so Point (5) is proved. Regarding Point (3), the partial differential of $(\mathcal{E}_{3,jk,i})_{1\leq i\leq 5}$ with respect to $(A_{jk}^+, C_{jk}^+, \nu_{jk}^+, A_{jk}^0, C_{jk}^0)$ is clearly an automorphism of $(\mathcal{W}_{\mathbb{R}}^{\geq 0})^3 \times \mathbb{R}^2$. Observe that $d\mathcal{E}_{3,jk,i}$ for $1 \leq i \leq 5$ do not involve the real variables θ_{jk} and θ_{kj} so $d\mathcal{E}_{3,jk}$ has block-triangular form and is suffices to compute the differential of the remaining two equations with respect to these variables:

$$d_{\theta_{jk},\theta_{kj}}\mathcal{E}_{3,jk,6} = \tfrac{1}{2}(-d\theta_{jk} + d\theta_{kj})$$

$$d_{\theta_{jk},\theta_{kj}}\mathcal{E}_{3,jk,7} = \tfrac{1}{2}(d\theta_{jk} + d\theta_{kj}).$$

Points (3) follows. $\square$

9. Solving all equations with the Implicit Function Theorem

There remains a few parameters that we have not used yet and that we can fix, namely: $r_{jk} = \tau_{jk}$ for $(j,k) \in E^+$, $\widehat{a}^0_{jk} = \tau_{jk}/2$ and $\theta^0_{jk} = \arg(u_{jk})$ for $(j,k) \in R$. Remembering that $q_{jk} = i\nu_{jk}$, we define

$$\begin{aligned}
\mathrm{x}_1 &= (\mathrm{x}_{1,jk})_{(j,k)\in E} \quad \mathcal{E}_1 = (\mathcal{E}_{1,jk})_{(j,k)\in E}\\
\mathrm{x}_2 &= (\mathrm{x}_{2,jk})_{(j,k)\in R} \quad \mathcal{E}_2 = (\mathcal{E}_{2,jk})_{(j,k)\in R}\\
\mathrm{x}_3 &= (\mathrm{x}_{3,jk})_{(j,k)\in E^+} \quad \mathcal{E}_3 = (\mathcal{E}_{3,jk})_{(j,k)\in E^+}\\
\mathrm{x}_4 &= (r_{kj}, \nu^0_{jk})_{(j,k)\in E^+} \quad \mathcal{E}_4 = (\mathcal{R}_{jk})_{(j,k)\in E^+}\\
\mathrm{x} &= (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) \quad \mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4).
\end{aligned}$$

Recall that the central value $\underline{\mathrm{x}}$ depends smoothly on the graph Γ (which has not yet been assumed to be balanced).

Proposition 16. *The partial differential of $\mathcal{E}(t,\mathrm{x})$ with respect to x at $(0,\underline{\mathrm{x}})$ is an automorphism. By the Implicit Function Theorem, for $t \geq 0$ in a neighhorhood of 0, there exists $\mathrm{x}(t,\Gamma)$, depending smoothly on t, $t\log t$ and the graph Γ, such that $\mathcal{E}(t,\mathrm{x}(t,\Gamma)) = 0$ and $\mathrm{x}(0,\Gamma) = \underline{\mathrm{x}}(\Gamma)$.*

Proof:

(1) By Propositions 11, 12 and 15, the partial differential of $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ with respect to $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3)$ has upper-triangular 3×3 block-form, with automorphisms on the diagonal, so is an automorphism.

(2) Let us prove that $L = d_{\mathrm{x}}\mathcal{E}(0,\underline{\mathrm{x}})$ is injective. Let $X \in \mathrm{Ker}(L)$. By Point (5) of Proposition 15, $dA_{jk}(X) = dC_{jk}(X) = 0$. By Point (5) of Proposition 11, $db_{jk}(X) = 0$. Differentiating Equation (33) and remembering that we fixed r_{jk} so $dr_{jk} = 0$, we obtain

$$d_{\mathrm{x}}\, \mathcal{R}_{jk}(0,\underline{\mathrm{x}}) = -\frac{1}{2} dr_{kj} + 2i\, db^0_{jk} + 2i\tau_{jk} d\nu^0_{jk}.$$

Hence $dr_{kj}(X) = d\nu^0_{jk}(X) = 0$, so $X_4 = 0$. Hence, by Point (1), $X = 0$.

(3) Since $\mathrm{x}_4, \mathcal{E}_4$ are in spaces of the same finite dimension, Points (1) and (2) imply that L is an automorphism by elementary linear algebra. By Point (2) of Proposition 13, $\mathcal{E}(t,\mathrm{x})$ is a smooth function of t, $t\log t$ and x, which (Definition 11) means that there exists a smooth function $\widetilde{\mathcal{E}}(t,s,\mathrm{x})$ such that $\mathcal{E}(t,\mathrm{x}) = \widetilde{\mathcal{E}}(t, t\log t, \mathrm{x})$. We apply the Implicit Function Theorem to $\widetilde{\mathcal{E}}$ at $(t,s,\mathrm{x}) = (0,0,\underline{\mathrm{x}}(\Gamma))$ and obtain a smooth function $\mathrm{x}(t,s,\Gamma)$ such that $\widetilde{\mathcal{E}}(t,s,\mathrm{x}(t,s,\Gamma)) = 0$. Specializing to $s = t\log t$, we obtain Proposition 16. $\square$

We are not done yet ; we still have to solve the equations $\mathcal{R}_j = 0$ and $\mathcal{L}_{jk} = 0$, where $\mathcal{R}_j$ is defined by Equation (27) and $\mathcal{L}_{jk}$ is defined by Equation (57). Define

$$\mathcal{F}(t,\Gamma) = \Big(\big(\mathcal{R}_j(t,\mathrm{x}(t,\Gamma))\big)_{j\in J}, \big(\mathcal{L}_{jk}(t,\mathrm{x}(t,\Gamma))\big)_{(j,k)\in E^+} \Big).$$

By Proposition 6 and since $\widetilde{P}_{jk}(0,\underline{\mathrm{x}}) = 0$, we have:

$$\mathcal{F}(0,\Gamma) = \Big(\big(\tfrac{1}{2}\,\overline{F_j(\Gamma)}\big)_{j\in J}, \big(1 - \tfrac{1}{2}\ell_{jk}(\Gamma)\big)_{(j,k)\in E^+} \Big).$$

By the Implicit Function Theorem, we obtain:

Proposition 17. *Assume that the central graph $\underline{\Gamma}$ has length-2 edges, is balanced and non-degenerate. Then for $t \geq 0$ small enough, there exists a deformation $\Gamma(t)$ of $\underline{\Gamma}$, depending smoothly on t and $t\log t$, such that $\Gamma(0) = \underline{\Gamma}$ and $\mathcal{F}(t,\Gamma(t)) = 0$.*

10. Geometry of the immersion

From now on, we assume that the parameter vector x is given by Proposition 16 and $\Gamma(t)$ is given by Proposition 17. We write $\mathrm{x}_t = \mathrm{x}(t, \Gamma(t))$ which is a smooth function of t and $t \log t$. To ease notation, we write $\overline{\Sigma}_t = \overline{\Sigma}_{t,\mathrm{x}_t}$ and $\xi_t = \xi_{t,\mathrm{x}_t}$. In the same way, we write $a_{jk,t}$, $p_{jk,t}$, etc... for the value of the parameters a_{jk}, p_{jk} at time t. We also write $\tau_{jk,t}$, $u_{jk,t}$, etc... for the quantities associated to $\Gamma(t)$, and $\tau_{jk} = \tau_{jk,0}$, $u_{jk} = u_{jk,0}$ for the quantities associated to the given graph $\Gamma(0)$.

We denote Σ_t the Riemann surface $\overline{\Sigma}_t$ minus the poles of ξ_t. Let $p : \widetilde{\Sigma}_t \to \Sigma_t$ be a universal cover. Recall that we have fixed an arbitrary $j_0 \in J$ and taken $z_0 = 1_{j_0}$ as base point. Choose an arbitrary $\widetilde{z}_0$ in the fiber $p^{-1}(z_0)$. Let Φ_t be the solution of $d\Phi_t = \Phi_t \xi_t$ on $\widetilde{\Sigma}_t$ with initial condition $\Phi_t(\widetilde{z}_0) = I_2$, $f_t = \mathrm{Sym}(\mathrm{Uni}(\Phi_t))$ the immersion given by the DPW method and $\widetilde{f}_t = \Psi \circ f_t$ where Ψ is the rigid motion given by Equation (10). Recall that $\overline{\Sigma}_t$ does not depend on λ, but Σ_t does, which is a problem as the DPW method requires a fixed Riemann surface. We address this issue in Section 10.2 using the results from [28] where the same problem already occured. At this point, all we know for sure is that f_t is a well defined immersion on Σ_t minus ε-neighborhoods of 0_{jk}, ∞_{jk} for $(j,k) \in E^+$ and u_{jk} for $(j,k) \in R$.

Fix a small ε_1 such that $0 < \varepsilon_1 \le \varepsilon/2$ and for $t > 0$ small enough, consider the following fixed compact subdomains of $\overline{\Sigma}_t$:

$$\Omega_{j,\varepsilon_1} = \overline{\mathbb{C}}_j \setminus \bigcup_{k \in E_j \cup R_j} D(u_{jk}, \varepsilon_1) \quad \text{for } j \in J \text{ (spherical parts)}$$

$$\Omega_{jk,\varepsilon_1} = \overline{\mathbb{C}}_{jk} \setminus D(\pm 1, \varepsilon_1) \quad \text{for } (j,k) \in E^+ \text{ (catenoidal parts).}$$

10.1. **Spherical parts.** Without loss of generality, we may assume by translating the graph that $v_{j_0} = 0$ so $V_{j_0} = 0$. Recall the definition of the gauge G_j in Section 7.1 which we now denote $G_{j,t}$ as it depends on t.

Proposition 18. *For $j \in J$ and $t > 0$:*

(1) *The potential ξ_t restricted to $\Omega_{j,\varepsilon_1} \setminus \{0_j, \infty_j\}$ depends C^1 on t.*
(2) *$\widetilde{f}_t(1_j) = v_{j,t} + e_1$.*
(3) *$\xi_t \cdot G_{j,t}$ is regular at 0_j and ∞_j, so f_t extends analytically to 0_j and ∞_j.*
(4) *As $t \to 0$, $\widetilde{f}_t - v_{j,t}$ converges on Ω_{j,ε_1} to the inverse stereographic projection $\pi^{-1} : \overline{\mathbb{C}} \to \mathbb{S}^2$. More precisely, we have*

$$\|\widetilde{f}_t - v_{j,t} - \pi^{-1}\|_{C^1(\Omega_{j,\varepsilon_1})} \le c\, t$$

for some uniform constant c (depending on ε_1) and the norm is computed for the spherical metric on the Riemann sphere.

Proof:

(1) Recall that $\mathrm{x}(t)$ is a smooth function of t and $t \log t$ so is not even differentiable at $t = 0$. However, assuming that Equation (24) holds, we have, for all values of the parameter x, $\xi_{0,\mathrm{x}} = \xi^S$ in Ω_{j,ε_1}, so $\xi_{0,\mathrm{x}}$ does not depend on x. By Proposition 24 in Appendix C, $\xi_t = \xi_{t,\mathrm{x}(t)}$, restricted to $\Omega_{j,\varepsilon_1} \setminus \{0_j, \infty_j\}$, extends to a C^1 function of t in a neighborhood of 0.
(2) Choose a path c from z_0 to 1_j on Σ_t and let $\widetilde{c}$ be the lift of c to $\widetilde{\Sigma}_t$ such that $\widetilde{c}(0) = \widetilde{z}_0$. Let $\widetilde{1}_j = \widetilde{c}(1) \in p^{-1}(1_j)$. Let $\widetilde{\Omega}_{j,\varepsilon_1}$ be the component of $p^{-1}(\Omega_{j,\varepsilon_1} \setminus \{0_j, \infty_j\})$ containing $\widetilde{1}_j$. Since $\xi_0 = \xi^S$ in $\overline{\mathbb{C}}_j$ we have

$$\Phi_0 = \Phi_0(\widetilde{1}_j)\Phi^S \quad \text{in } \widetilde{\Omega}_{j,\varepsilon_1}. \tag{58}$$

By Equation (39), we have

$$\begin{cases} \Phi_t(\widetilde{1}_j) \in \Lambda SU(2) \\ \Phi_t(\widetilde{1}_j) \mid_{\lambda=1} = \pm I_2 \\ \Phi_t(\widetilde{1}_j)^{-1} \frac{\partial}{\partial \lambda} \Phi_t(\widetilde{1}_j) \mid_{\lambda=1} = i\, V_{j,t}. \end{cases} \tag{59}$$

By the Sym-Bobenko formula (4) and Equation (41),

$$f_t(1_j) = 2V_{j,t} \sim (0, -\mathrm{Im}(v_{j,t}), -\mathrm{Re}(v_{j,t})).$$

$$\widetilde{f}_t(1_j) = \Psi(f_t(1_j)) = (1 + \mathrm{Re}(v_{j,t}), \mathrm{Im}(v_{j,t}, 0).$$

(3) To prove Point (3), we apply Theorem 4 in Appendix A to the potential $\widehat{\xi}_t = \xi_t \cdot G_{j,t}$. Let ℓ be the maximum of $E_j \cup R_j$ for the order $\prec$. The path $\delta_{j\ell}$ bounds a disk-type domain in Ω_{j,ε_1} containing 0_j and ∞_j and not containing -1. The potential $\widehat{\xi}_t$ satisfies Hypothesis (1) to (3) of Theorem 4 in Ω thanks to Propositions 5, 6 and 17. By Equation (34), Φ_t solves the Monodromy Problem on $\delta_{j\ell}$. At $t = 0$ we have by Equation (23) $x_j = 1$ and $y_j = 0$ so $G_{j,0} = G^S$. Hence

$$\widehat{\xi}_0 = \xi^S \cdot G^S = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} \frac{dz}{2(z+1)^2}.$$

Let $\widehat{\Phi}_t = \Phi_0(\widetilde{1}_j)^{-1}\Phi_t G_{j,t}$. Since $\Phi_0(\widetilde{1}_j) \in \Lambda SU(2)$, $\widehat{\Phi}_t$ solves the Monodromy Problem on $\delta_{j\ell}$ and

$$\widehat{\Phi}_0 = \Phi^S G^S = \begin{pmatrix} 2 & \frac{z-1}{2\lambda(z+1)} \\ 0 & 1/2 \end{pmatrix}.$$

Theorem 4 tells us that $\widehat{\xi}_t$ is holomorphic at 0_j. Finally $\widehat{\beta}_0(0_j) = dz/2$ so $\widehat{\beta}_t^0(0_j) \neq 0$ for t small enough so f_t is regular at 0_j. Regularity at ∞_j follows by σ-symmetry.

(4) Let $\check{\Phi}_t$ be the solution of $d\check{\Phi}_t = \check{\Phi}_t\xi_t$ with initial condition $\check{\Phi}(\widetilde{1}_t) = I_2$ and $\check{f}_t = \mathrm{Sym}(\mathrm{Uni}(\check{\Phi}))$. By Point (1) and standard ODE theory, $\check{\Phi}_t$ is a C^1 function of t in a neighborhood of 0 and $z \in \Omega_{j,\varepsilon_1} \setminus \{0_j, \infty_j\}$. Since Iwasawa decomposition is a diffeomorphism (Theorem 2), $\mathrm{Uni}(\check{\Phi}_t)$ and $\mathrm{Pos}(\check{\Phi}_t)$ are C^1, so by Equation (6), $d\check{f}_t$ is C^1. Let K be a compact subset of $\overline{\Omega}_{j,\varepsilon_1} \setminus \{0_j, \infty_j\}$. By the mean value inequality,

$$\|\check{f}_t(z) - \check{f}_t(1_j) - \check{f}_0(z) + \check{f}_0(1_j)\| \leq C(K)t \quad \text{for } z \in K.$$

Since $\Phi_t(\widetilde{1}_j) \mid_{\lambda=1} = I_2$, f_t and $\check{f}_t$ differ by a translation. Also $\xi_0 = \xi^S$ so $\check{f}_0 = f^S$. Hence

$$\|f_t(z) - f_t(1_j) - f^S(z) + f^S(1)\| \leq C(K)t.$$

By Point (2) and Equation (11) we obtain

$$\|\widetilde{f}_t(z) - v_{j,t} - \pi^{-1}(z)\| \leq C(K)t \quad \text{for } z \in K.$$

This estimate is extended to neighborhoods of 0_j and ∞_j using the gauge $G_{j,t}$. □

10.2. **Delaunay ends.** For $p \in \mathbb{C}$, $D^*(p, r)$ denotes the punctured disk $0 < |z - p| < r$.

Proposition 19. *There exists $\varepsilon_2 > 0$ such that for $(j, k) \in R$ and $t > 0$ small enough:*

(1) *f_t extends analytically to $D^*(p^0_{jk,t}, \varepsilon_2)$.*
(2) *f_t has a Delaunay end of weight $\simeq 2\pi t\tau_{jk}$ at $p^0_{jk,t}$.*
(3) *The axis of the Delaunay end of $\widetilde{f}_t$ at $p^0_{jk,t}$ converges to the half-line $v_j + \mathbb{R}^+ u_{jk}$ as $t \to 0$.*
(4) *If $\tau_{jk} > 0$, then $f_t(D^*(p^0_{jk,t}, \varepsilon_2))$ is embedded.*

Proof: These facts are proved in [28] in a similar situation, using general results about Delaunay ends from [16] and [20]. The potential in [28] has the form

$$\begin{pmatrix} 0 & \lambda^{-1}dz \\ t(\lambda-1)^2\omega_t & 0 \end{pmatrix}$$

where ω_t has double poles. We gauge our potential to a similar form so we can apply the results of [28]. Fix $(j,k)\in R$. Recall that α_t, β_t are holomorphic at $p_{jk,t}$ and γ_t has a double pole with principal part

$$\gamma_t = t(\lambda-1)^2\left(\frac{\widehat{a}_{jk,t}p_{jk,t}dz}{(z-p_{jk,t})^2}+\frac{i\,\widehat{b}_{jk,t}dz}{z-p_{jk,t}}+O(1)dz\right)$$

where $O(1)$ means a holomorphic function in a neighborhood of $p_{jk,t}$. Define $\kappa_t\in\mathcal{W}^{\geq 0}$ by $\kappa_t = p_{jk,t}\beta_t(p_{jk,t})/dz$. At $t=0$, we have $\beta_0=\frac{dz}{2z}$ so $\kappa_0=1/2$. Consider the gauge

$$G_t=\begin{pmatrix} \sqrt{\frac{\kappa_t}{z}} & 0 \\ \frac{\lambda}{2\sqrt{\kappa_t z}} & \sqrt{\frac{z}{\kappa_t}} \end{pmatrix}.$$

A computation gives

$$\widehat{\xi}_t := \xi_t\cdot G_t = \begin{pmatrix} \alpha_t+\frac{\beta_t}{2\kappa_t}-\frac{dz}{2z} & \frac{z\beta_t}{\lambda\kappa_t} \\ -\frac{\lambda\alpha_t}{z}-\frac{\lambda\beta_t}{4\kappa_t z}+\frac{\kappa_t\gamma_t}{z} & -\alpha_t-\frac{\beta_t}{2\kappa_t}+\frac{dz}{2z} \end{pmatrix}.$$

Thanks to our choice of κ_t and given the principal part of γ_t, $\widehat{\xi}_t$ has the form

$$\widehat{\xi}_t = \begin{pmatrix} 0 & \lambda^{-1}dz \\ t(\lambda-1)^2\omega_t & 0 \end{pmatrix} + \begin{pmatrix} O(1) & O(z-p_{jk,t}) \\ O(1) & O(1) \end{pmatrix} \tag{60}$$

with

$$\omega_t=\kappa_t\left(\frac{\widehat{a}_{jk,t}dz}{(z-p_{jk,t})^2}+\frac{(i\widehat{b}_{jk,t}-\widehat{a}_{jk,t})dz}{p_{jk,t}(z-p_{jk,t})}\right).$$

The gauged potential $\widehat{\xi}_t$ now has the same form as in [28] up to a holomorphic term which is of no consequence (see Remark 11 below). By Proposition 4 in [28], f_t extends analytically to $D^*(p^0_{jk,t},\varepsilon_2)$, $\kappa_t\widehat{a}_{jk,t}$ is a real constant and f_t has a Delaunay end of weight $8\pi t\kappa_t\widehat{a}_{jk,t}$ at $p^0_{jk,t}$. Since $\kappa_0\widehat{a}_{jk,0}=\tau_{jk}/4$, Point (2) follows. Let $\widehat{\Phi}_t=\Phi_0(\widetilde{1}_j)^{-1}\Phi_t G_t$. At $t=0$ we have by Equation (58)

$$\widehat{\Phi}_0(z)=\Phi^S(z)G_0(z)=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \lambda^{-1}(z-1) \\ \lambda & z+1 \end{pmatrix} = H\begin{pmatrix} 1 & \lambda^{-1}z \\ 0 & 1 \end{pmatrix},\quad H=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\lambda^{-1} \\ \lambda & 1 \end{pmatrix}.$$

Let $\check{\Phi}_t = H^{-1}\widehat{\Phi}_t$ and $\check{f}_t$ be the corresponding immersion. Then $\check{\Phi}_0$ has the same value as Φ_0 in [28]. By Proposition 5 in [28], the axis of the Delaunay end of $\check{f}_t$ at $p^0_{jk,t}$ converges to the half-line through $(0,0,1)$ spanned by $-u_{jk}$. (The signs in Proposition 5 are actually opposite, but this is because we have the opposite Sym-Bobenko formula in [28].) Applying the isometries represented by H, $\Phi_0(\widetilde{1}_j)$ and the rigid motion Ψ, we obtain Point (3). Point (4) is proved in Proposition 6 in [28].

Remark 11. The proof of Proposition 4 in [28] uses a gauge of the form

$$G=\begin{pmatrix} \frac{\sqrt{w}}{k} & 0 \\ \frac{-\lambda}{2k\sqrt{w}} & \frac{k}{\sqrt{w}} \end{pmatrix}\qquad \text{with } w=z-p_{jk,t} \text{ and } k\in\mathcal{W}^{\geq 0}.$$

Then

$$\widehat{\xi}_t\cdot G=\begin{pmatrix} \widehat{\alpha}_t-\frac{\widehat{\beta}_t}{2w}+\frac{dw}{2w} & \frac{k^2\widehat{\beta}_t}{\lambda w} \\ \frac{\lambda\widehat{\alpha}_t}{k^2}-\frac{\lambda\widehat{\beta}_t}{4k^2w}+\frac{w\widehat{\gamma}_t}{k^2}+\frac{\lambda\,dw}{2k^2w} & -\widehat{\alpha}_t+\frac{\beta_t}{2w}-\frac{dw}{2w} \end{pmatrix}.$$

What only matters is the residue of $\widehat{\xi}_t \cdot G$ at $w=0$. So the second term in the right-hand side of (60) can be neglected because its $(1,2)$ entry has a zero at $w=0$ and the other entries are holomorphic.

10.3. **Catenoidal parts.** Recall from Section 7.2 the definition of the complex coordinate w_{jk} on $\overline{\mathbb{C}}_{jk}$ and the gauge G_{jk}, which we write respectively $w_{jk,t}$ and $G_{jk,t}$ as they now depend on t. We denote $\widehat{\xi}_{jk,t} = (w_{jk,t}^{-1})^* \xi_t \cdot G_{jk,t}$. We cannot use $z=1$ as base point in $\overline{\mathbb{C}}_{jk}$ so we use instead the point i_{jk} defined as $z=i$ if $\tau_{jk}>0$ and $z=-i$ if $\tau_{jk}<0$. This point lies on the path Γ_{jk}.

Proposition 20. *For $(j,k)\in E^+$ and $t>0$ small enough:*

1. *The potential $\widehat{\xi}_{jk,t}$ is regular at $w=0$. Consequently, the immersion f_t extends analytically to a neighborhood of 0_{jk} and ∞_{jk}.*
2. *The potential ξ_t is regular on Σ_t, so f_t is a regular immersion.*
3. *The blow-up $t^{-1}(\widetilde{f}_t - \widetilde{f}_t(i_{jk})))$ converges on $\Omega_{jk,\varepsilon_1}$ as $t\to 0$ to a minimal catenoidal immersion from $\mathbb{C}\setminus\{\pm 1\}$ to $\mathbb{R}^3$. The limit catenoid has waist radius $|\tau_{jk}|$ and its axis, oriented from the end at $z=1$ to the end at $z=-1$, is a line parallel to u_{jk} and oriented by $\tau_{jk}u_{jk}$. The convergence is for the C^1 norm.*

Proof: fix $(j,k)\in E^+$.

1. We start by computing Φ_0 in $\Omega_{jk,\varepsilon_1}$. Split the path Γ_{jk} as $\Gamma_{jk} = \Gamma_{jk1}\Gamma_{jk2}$ with $\Gamma_{jk1}(1) = \Gamma_{jk2}(0) = i_{jk}$. Consider the lift of Γ_{jk1} to $\widetilde{\Sigma}_t$ starting at $\widetilde{1}_j$ and let $\widetilde{\imath}_{jk}$ be its endpoint. Consider the lift of Γ_{jk2} starting at $\widetilde{\imath}_{jk}$ and let $\widetilde{1}_k$ be its endpoint. Let $\widetilde{\Omega}_{jk,\varepsilon_1}$ be the component of $p^{-1}(\Omega_{jk,\varepsilon_1}\cap\Sigma_t)$ which contains $\widetilde{\imath}_{jk}$. By Theorem 5, $\Phi_t(\widetilde{\imath}_{jk})$ extends to a smooth function of t and $t\log t$. Moreover, since $\xi_0 = \xi^C$ in $\Omega_{jk,\varepsilon_1}$, we have $\Phi_0 = M\Phi^C$ for some matrix M which is determined by the fact that Φ_0 is continuous at the nodes (see Remark 14). This gives by Equation (58)

$$\Phi_0(z) = \Phi_0(\widetilde{1}_j)\Phi^S(u_{jk})\Phi^C(z) = \Phi_0(\widetilde{1}_k)\Phi^S(u_{kj})\Phi^C(-1)^{-1}\Phi^C(z) \quad \text{in } \widetilde{\Omega}_{jk,\varepsilon_1}. \tag{61}$$

2. The proof of Point (1) is essentially the same as the proof of Point (3) of Proposition 18. We apply the dual version of Theorem 4, Corollary 1 in Appendix A. Observe that

$$\Gamma_{jk1}^{-1}\gamma_{jk}\Gamma_{jk1}\in\pi_1(\Sigma_t, i_{jk}) \tag{62}$$

is homotopic to a loop δ_1 contained in $\Omega_{jk,\varepsilon_1}$ going around 1 in the clockwise direction, and

$$\Gamma_{jk2}\gamma_{kj}\Gamma_{jk2}^{-1}\in\pi_1(\Sigma_t, i_{jk}) \tag{63}$$

is homotopic to a loop δ_2 contained in $\Omega_{jk,\varepsilon_1}$ going around -1 in the clockwise direction. The product of the loops (62) and (63) is a reparametrization (changing the base point) of

$$\gamma_{jk}\Gamma_{jk}\gamma_{kj}\Gamma_{jk}^{-1}. \tag{64}$$

The Monodromy Problem for Φ_t on the loop (64) is solved so it is also solved on $\delta_1\delta_2$. We now make the change of variable $w = w_{jk,t}(z)$. The path $w_{jk,t}(\delta_1\delta_2)$ bounds a disk-type domain in $\overline{\mathbb{C}}\setminus\{\pm 1\}$ containing 0 and ∞. The potential $\widehat{\xi}_t$ satisfies Hypothesis (1) to (3) of Corollary 1 thanks to Propositions 7, 8 and 16. Let $\widehat{\Phi}_t = (w_{jk,t}^{-1})^*\Phi_t G_{jk,t}$. At $t=0$, we have $w_{jk,0}(z)=z$ and by Equation (29) $x_{jk}=1$, $y_{jk}=0$ so

$$G_{jk,0} = \begin{pmatrix} \frac{\sqrt{z}}{1+z} & \frac{1-z}{\sqrt{z}} \\ 0 & \frac{1+z}{\sqrt{z}} \end{pmatrix}$$

$$\widehat{\xi}_0 = \xi^C\cdot G_{jk,0} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\frac{dz}{2(z+1)^2}.$$

Using Equation (61), we have

$$\widehat{\Phi}_0 = \Phi_0(\widetilde{1}_j)\Phi^S(u_{jk})\Phi^C G_{jk,0} = \Phi_0(\widetilde{1}_j)\Phi^S(u_{jk})\begin{pmatrix} 1/2 & 0 \\ \frac{z-1}{2(z+1)} & 2 \end{pmatrix}$$

where $\Phi_0(\widetilde{1}_j)\Phi^S(u_{jk}) \in \Lambda SU(2)$. By Corollary 1, $\widehat{\xi}_t$ is holomorphic at 0.

Remark 12. To deal with the fact that $\widehat{\xi}_t$ is not C^1 with respect to t, we write $t = \exp(-1/s^2)$, so $\widehat{\xi}_{t(s)}$ extends to a smooth function of s in a neighborhood of 0, and use s as the time parameter when applying Corollary 1.

(3) By Equation (32), since $\widehat{\beta}_t$ is holomorphic at $w=0$, $\widetilde{\beta}_t^0$ has a zero of multiplicity at least one at $w=0$. So β_t^0 has a zero of multiplicity at least one at $z = q^0_{jk,t}$ and at $z = \sigma(q^0_{jk,t})$ by symmetry, for a total of $2\,\mathrm{card}(E^+)$ zeros. It has simple poles at 0_j and ∞_j for $j \in J$. By elementary topology, the genus of $\overline{\Sigma}_t$ is $g = \mathrm{card}(E^+) - \mathrm{card}(J) + 1$. Hence the number of zeros of β_t^0, counting multiplicities, is equal to

$$\#\text{poles} + 2g - 2 = 2\,\mathrm{card}(J) + 2g - 2 = 2\,\mathrm{card}(E^+).$$

So the zeros at $q^0_{jk,t}$ and $\sigma(q^0_{jk,t})$ are simple and β_t^0 has no other zero. This proves Point (2), and yields that $\widehat{\beta}_t^0$ does not vanish at $w=0$, so completes the proof of Point (1).

(4) To prove Point (3), we use Theorem 4 in [29]. One technical issue is that this theorem requires a C^1 family of potentials ξ_t and we do not have that regularity. This problem is solved as follows. Forget for a moment that the parameter x has been determined as a smooth function of t and $\log t$ and consider the potential $\xi_{t,\mathrm{x}}$, only assuming that the parameter B_{jk} is given by Equation (30). Consider the gauged potential

$$\check{\xi}_{t,\mathrm{x}} = \xi_{t,\mathrm{x}} \cdot \check{G}_{\mathrm{x}} \quad \text{with} \quad \check{G}_{\mathrm{x}} = \frac{1}{\sqrt{2C_{jk}}}\begin{pmatrix} 1 & 2i\,A_{jk} \\ 0 & 2C_{jk} \end{pmatrix}.$$

Then at $t=0$ we have in $\Omega_{jk,\varepsilon_1}$, using Proposition 3 and Equation (30):

$$\check{\xi}_{0,\mathrm{x}} = \eta_{0,\mathrm{x}} \cdot \check{G}_{\mathrm{x}} = \begin{pmatrix} 0 & 2(B_{jk}C_{jk} - A_{jk}^2) \\ 1/2 & 0 \end{pmatrix}\frac{dz}{z} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & \end{pmatrix}\frac{dz}{z} = \xi^C.$$

Since this does not depend on x, Proposition 24 in Appendix C ensures that $\check{\xi}_t = \check{\xi}_{t,\mathrm{x}(t)}$ extends to a C^1 function in a neighborhood of $t=0$. Moreover

$$\frac{d}{dt}\check{\xi}_t = \frac{\partial}{\partial t}\check{\xi}_{t,\mathrm{x}}\mid_{(t,\mathrm{x})=(0,\mathrm{x}(0))} = \frac{\partial}{\partial t}\xi_{t,\underline{\mathrm{x}}}\mid_{t=0}.$$

Define in $\widetilde{\Omega}_{jk,\varepsilon_1}$

$$\check{\Phi}_t = \check{H}_{jk,t}\Phi_t\check{G}_{\mathrm{x}(t)} \quad \text{with} \quad \check{H}_{jk,t} = \Phi^C(\widetilde{\imath}_{jk})\mathrm{Uni}(\Phi_t(\widetilde{\imath}_{jk}))^{-1}.$$

At $t=0$, we have $\check{G}_{\mathrm{x}(0)} = I_2$ and by Equation (61)

$$\check{H}_{jk,0} = \Phi^C(\widetilde{\imath}_{jk})\Phi_0(\widetilde{\imath}_{jk})^{-1} = \left(\Phi_0(\widetilde{1}_j)\Phi^S(u_{jk})\right)^{-1}. \tag{65}$$

Hence $\check{\Phi}_0 = \Phi^C$ in $\widetilde{\Omega}_{jk,\varepsilon_1}$. Let $\check{f}_t = \mathrm{Sym}(\mathrm{Uni}(\check{\Phi}_t))$. By Theorem 4 in [29], $t^{-1}\check{f}_t$ converges to a minimal immersion with Weierstrass data

$$g = -\frac{\check{\Phi}_{0;11}}{\check{\Phi}_{0;21}} = \frac{z+1}{1-z}$$

$$\omega = 4(\check{\Phi}_{0;21})^2\frac{\partial}{\partial t}\check{\beta}_t^0\mid_{t=0} = \frac{(z-1)^2}{z}\frac{\tau_{jk}}{2}\left(\frac{dz}{(z-1)^2} - \frac{dz}{(z+1)^2}\right) = \frac{2\tau_{jk}\,dz}{(z+1)^2}$$

using Proposition 4. With the change of variable $w = (z+1)/(1-z)$ we obtain $g = w$ and $\omega = \tau_{jk} dw/w^2$. This is the Weierstrass data of a catenoid with neck-size $|\tau_{jk}|$ and vertical axis (from the end at $w = \infty$ to $w = 0$) oriented by $-\tau_{jk} e_3$. Let $h_{jk,t}$ be the rigid motion represented by $\check{H}_{jk,t}$ and $\vec{h}_{jk,t}$ its linear part, where the action is given by (9). We have $\check{f}_t = h_{jk,t} \circ f_t$. At $t = 0$, we have by Equation (65) $\check{H}_{jk,0} \mid_{\lambda=1} = \Phi^S(u_{jk})^{-1} \mid_{\lambda=1}$. So by Equation (5), $\vec{h}_{jk,0}$ maps $N^S(u_{jk})$ to e_3. This means that $t^{-1}(f_t - f_t(i_{jk}))$ converges to a catenoid with axis (from the end at $z = 1$ to $z = -1$) oriented by $-\tau_{jk} N^S(u_{jk})$. We have

$$\vec{\Psi}(N^S(u_{jk})) = (-\mathrm{Re}(u_{jk}), -\mathrm{Im}(u_{jk}), 0) \sim -u_{jk}$$

so $t^{-1}(\widetilde{f}_t - \widetilde{f}_t(i_{jk}))$ converges to a catenoid with axis oriented by $\tau_{jk} u_{jk}$. The convergence is on compact subsets of $\Omega_{jk,\varepsilon_1} \setminus \{0_{jk}, \infty_{jk}\}$. It is extended to neighborhoods of 0_{jk} and ∞_{jk} using the gauge $G_{jk,t}$. □

10.4. **Edge-length estimate.** Recall that $\ell_{jk,t} = \|v_{k,t} - v_{j,t}\|$ is the length of the edge (j,k) on Γ_t.

Proposition 21. *As $t \to 0$, we have for $(j,k) \in E^+$*

$$\ell_{jk,t} = 2 - 2\,\tau_{jk}\, t \log t + O(t). \tag{66}$$

$$\widetilde{f}_t(i_{jk}) = \tfrac{1}{2}(v_{j,t} + v_{k,t}) + O(t) \tag{67}$$

Remark 13.
1. Equation (66) estimates how much the spheres centered at v_j and v_k move away from each other if $\tau_{jk} > 0$ (or toward each other if $\tau_{jk} < 0$) to fit in a catenoidal neck of size $\simeq \tau_{jk} t$. It is in agreement with the half-period of a Delaunay surface of necksize $\tau_{jk} t$ which is known to have the asymptotic (66) as $t \to 0$ (see for example Proposition 7 in [18] – a scaling of 1/2 must be applied because the mean curvature is the trace of the fundamental form in that paper).
2. Equation (67) tells us that the waist of the catenoidal neck is centered at the middle of $v_{j,t}$, $v_{k,t}$, up to an $O(t)$ term.

Proof: forget for a moment that x is determined as a function of t. We first compute the term of order $t\log t$ in $\mathcal{P}(\xi_{t,\mathrm{x}}, \Gamma_{jk})$. Recall that the only terms where a $t \log t$ appears are those corresponding to path numbers (2) and (4) in the definition of β_{jk}. To estimate the term corresponding to path number (2), we use Point (3) of Theorem 5 where γ denotes the circle $|z_{jk}| = \varepsilon$. We have

$$\mathcal{P}(\xi_{t,\mathrm{x}}, \gamma) = \mathcal{P}(\xi_{t,\mathrm{x}}, 1_j, z_{jk} = \varepsilon')^{-1} \mathcal{P}(\xi_{t,\mathrm{x}}, \gamma_{jk}) \mathcal{P}(\xi_{t,\mathrm{x}}, 1_j, z_{jk} = \varepsilon')$$

Using Equation (52)

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{P}(\xi_{t,\mathrm{x}}, \gamma) \mid_{t=0} = \Phi^S(z_{jk} = \varepsilon')^{-1} \frac{\partial}{\partial t}\mathcal{P}(\xi_{t,\mathrm{x}}, \gamma_{jk}) \mid_{t=0} \Phi^S(z_{jk} = \varepsilon')\\
&= \quad \Phi^S(z_{jk} = \varepsilon')^{-1} \Phi^S(p_{jk}) \widetilde{M}_{jk}(0, \mathrm{x}) \Phi^S(p_{jk})^{-1} \Phi^S(z_{jk} = \varepsilon')\\
&= \quad 2\pi i \tau_{jk} \frac{(\lambda-1)^2}{4\lambda} \Phi^S(z_{jk} = \varepsilon')^{-1} \Phi^S(u_{jk}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi^S(u_{jk})^{-1} \Phi^S(z_{jk} = \varepsilon') + O(\mathrm{x} - \underline{\mathrm{x}}).
\end{aligned}$$

By Theorem 5, the principal solution of $\xi_{t,\mathrm{x}}$ on path number (2) is equal to

$$\begin{aligned}
&\left[I_2 + \tau_{jk} t \log t \frac{(\lambda-1)^2}{4\lambda} \Phi^S(z_{jk} = \varepsilon')^{-1} \Phi^S(u_{jk}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi^S(u_{jk})^{-1} \Phi^S(z_{jk} = \varepsilon') \right]\\
&\times \quad \mathcal{P}(\xi_{0,\mathrm{x}}, z_{jk} = \varepsilon', z_{jk} = t_{jk}/\varepsilon') + O(t) + t \log t\, O(\mathrm{x} - \underline{\mathrm{x}}).
\end{aligned}$$

By the same argument, the principal solution of $\xi_{t,\mathrm{x}}$ on path number (4) is equal to

$$\left[I_2-\tau_{jk}t\log t\frac{(\lambda-1)^2}{4\lambda}\Phi^C(z'_{kj}=-\varepsilon')^{-1}\Phi^C(-1)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\Phi^C(-1)^{-1}\Phi^C(z'_{kj}=-\varepsilon')\right]$$
$$\times\quad \mathcal{P}(\xi_{0,\mathrm{x}},z'_{kj}=-\varepsilon',z'_{kj}=-t_{kj}/\varepsilon')+O(t)+t\log t\,O(\mathrm{x}-\underline{\mathrm{x}}).$$

The computation in the proof of Point (2) of Proposition 13 gives after simplification

$$\mathcal{P}(\xi_{t,\mathrm{x}},\Gamma_{jk})=\mathcal{P}(\xi_{0,\mathrm{x}},\Gamma_{jk})+\tau_{jk}t\log t\frac{(\lambda-1)^2}{4\lambda}\Phi^S(u_{jk})\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix},\Phi^C(-1)\right]\Phi^S(u_{kj})^{-1}+O(t)+t\log t\,O(\mathrm{x}-\underline{\mathrm{x}}).$$

Recalling the definition of P_{jk} and $\widetilde{P}_{jk}$ from Section 8.6, we obtain

$$P_{jk}(t,\mathrm{x})=P_{jk}(0,\mathrm{x})+\tau_{jk}t\log t\frac{(\lambda-1)^2}{4\lambda}\left[\begin{pmatrix}1&0\\0&-1\end{pmatrix},\Phi^C(-1)\right]\Phi^S(u_{kj})^{-1}\Phi^S(u_{jk})+O(t)+t\log t\,O(\mathrm{x}-\underline{\mathrm{x}}).$$

$$\widetilde{P}_{jk}(t,\mathrm{x})=\widetilde{P}_{jk}(0,\mathrm{x})+\tau_{jk}t\log t\frac{(\lambda-1)^2}{4\lambda}\begin{pmatrix}2&0\\0&-2\end{pmatrix}+O(t)+t\log t\,O(\mathrm{x}-\underline{\mathrm{x}})+O((\mathrm{x}-\underline{\mathrm{x}})^2).$$

We substitute the value $\mathrm{x}(t)=\mathrm{x}(t,\Gamma)$ given by Proposition 16. (At this point, the graph Γ is fixed.) Recalling that $\mathrm{x}(t)=\underline{\mathrm{x}}+O(t\log t)$ and $\widetilde{P}_{jk}(0,\underline{\mathrm{x}})=0$, we obtain

$$\widetilde{P}_{jk}(t,\mathrm{x}(t))=d_{\mathrm{x}}\widetilde{P}_{jk}(0,\underline{\mathrm{x}})(\mathrm{x}(t)-\underline{\mathrm{x}})+\tau_{jk}t\log t\frac{(\lambda-1)^2}{2\lambda}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+O(t). \tag{68}$$

Write $\delta\mathrm{x}(t)=\mathrm{x}(t)-\underline{\mathrm{x}}$ and extend this notation to all parameters. Recalling from the proof of Proposition 15 the formula for $d_{\mathrm{x}}\widetilde{P}_{jk}(0,\underline{\mathrm{x}})$ and the definition of $\mathcal{F}_{jk}$, we obtain

$$\mathcal{F}^+_{jk}(t,\mathrm{x}(t))=-2\,\delta C^+_{jk}(t)+\lambda\tau_{jk}t\log t+O(t)=0$$
$$\mathcal{F}^0_{jk}(t,\mathrm{x}(t))=-4\,\delta C^0_{jk}(t)-2\tau_{jk}t\log t+O(t)=0$$

which gives

$$\delta C_{jk}(t)=\frac{\tau_{jk}}{2}t\log t\,(\lambda-1)+O(t). \tag{69}$$

The definition of $\mathcal{L}_{jk}$ and Equation (68) give (recalling that τ_{jk} and ℓ_{jk} depend on the graph Γ)

$$\mathcal{L}_{jk}(t,\mathrm{x}(t,\Gamma))=\frac{\partial}{\partial\lambda}(-2\delta C_{jk}(t,\Gamma))\mid_{\lambda=1}-\frac{\ell_{jk}(\Gamma)-2}{2}=-\tau_{jk}(\Gamma)t\log t+O(t)-\frac{\ell_{jk}(\Gamma)-2}{2}.$$

By Proposition 17, the graph Γ_t satisfies $\mathcal{L}_{jk}(t,\mathrm{x}(t,\Gamma_t))=0$, and this gives Point (1) of Proposition 21.

Since the $t\log t$ factor in Equation (68) is diagonal, the resolution of the remaining equations of the system $\mathcal{E}_{3,jk}(t,\mathrm{x}(t))=0$, which only involve the off-diagonal part of $\widetilde{P}_{jk}(t,\mathrm{x}(t))$, gives

$$\delta A_{jk}(t)=O(t),\quad \delta q^+_{jk}(t)=O(t),\quad \delta\theta_{jk}(t)=O(t)\quad\text{and}\quad \delta\theta_{kj}(t)=O(t).$$

By Point (5) of Proposition 11, we obtain $\delta b_{jk}(t)=O(t)$. Finally, the resolution of $\mathcal{R}_{jk}(t,\mathrm{x}(t))=0$ gives $\delta r_{kj}(t)=O(t)$ and $\delta q^0_{jk}(t)=O(t)$ so $\delta q_{jk}(t)=O(t)$.

Recall that Γ_{jk1} denotes the first half of the path Γ_{jk}, from 1_j to i_{jk}. By a computation similar to the above we have

$$\mathcal{P}(\xi_{t,\mathrm{x}},\Gamma_{jk1})=\mathcal{P}(\xi_{0,\mathrm{x}},\Gamma_{jk1})+\tau_{jk}t\log t\frac{(\lambda-1)^2}{4\lambda}\Phi^S(u_{jk})\begin{pmatrix}1&0\\0&-1\end{pmatrix}\Phi^C(i_{jk})+O(t)+t\log t\,O(\mathrm{x}-\underline{\mathrm{x}})$$

$$\mathcal{P}(\xi_{0,\mathrm{x}},\Gamma_{jk1})=\Phi^S(p_{jk})\exp\left(M_{jk}\int_1^{i_{jk}}\omega_{q_{jk}}\right)$$

$$\frac{\partial}{\partial C_{jk}}\mathcal{P}(\xi_{0,\mathrm{x}},\Gamma_{jk1})=\Phi^S(u_{jk})\frac{\partial}{\partial C_{jk}}\exp\left(M_{jk}\pi\,i_{jk}/2\right)=\sqrt{2}\,i_{jk}\Phi^S(u_{jk})\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

We substitute $\mathrm{x} = \mathrm{x}(t)$. Using Equation (69) and that δq_{jk}, δA_{jk}, $\delta\theta_{jk}$ are $O(t)$, we obtain

$$\mathcal{P}(\xi_{t,\mathrm{x}(t)},\Gamma_{jk1}) = \Phi^S(u_{jk})\Phi^C(i_{jk})\big(I_2+\tau_{jk}t\log t\, Q_{jk}+O(t)\big)$$

with

$$\begin{aligned} Q_{jk} &= \frac{\sqrt{2}}{2} i_{jk}(\lambda-1)\Phi^C(i_{jk})^{-1}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix} + \frac{(\lambda-1)^2}{4\lambda}\Phi^C(i_{jk})^{-1}\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\Phi^C(i_{jk})\\ &= \frac{\lambda-1}{2}\begin{pmatrix} 1 & -i_{jk}\\ i_{jk} & -1\end{pmatrix} + \frac{(\lambda-1)^2}{4}\begin{pmatrix} 0 & i_{jk}\\ -i_{jk} & 0\end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\lambda-1) & \frac{i_{jk}}{4}(\lambda^{-1}-\lambda)\\ \frac{i_{jk}}{4}(\lambda-\lambda^{-1}) & \frac{1}{2}(1-\lambda)\end{pmatrix}\end{aligned}$$

The differential of Iwasawa decomposition at the identity is the projection on the factors of the decomposition of the Lie algebra $\Lambda\mathfrak{sl}(2,\mathbb{C})$ as $\Lambda\mathfrak{su}(2)\oplus\Lambda\mathfrak{sl}^+_{\mathbb{R}}(2,\mathbb{C})$. The matrix Q_{jk} decomposes as

$$Q_{jk} = \frac{i_{jk}}{4}\begin{pmatrix} 0 & \lambda^{-1}-\lambda\\ \lambda-\lambda^{-1} & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix} \lambda-1 & 0\\ 0 & 1-\lambda\end{pmatrix} \in \Lambda\mathfrak{su}(2)\oplus\Lambda\mathfrak{sl}^+_{\mathbb{R}}(2,\mathbb{C}).$$

Hence

$$\mathrm{Uni}(\mathcal{P}(\xi_{t,\mathrm{x}(t)},\Gamma_{jk1})) = \Phi^S(u_{jk})\Phi^C(i_{jk})\left[I_2 + \frac{\tau_{jk}\, i_{jk}}{4} t\log t\begin{pmatrix} 0 & \lambda^{-1}-\lambda\\ \lambda-\lambda^{-1} & 0\end{pmatrix} + O(t)\right].$$

Finally the Sym Bobenko formula (4) gives

$$\begin{aligned} &\mathrm{Sym}(\mathrm{Uni}(\mathcal{P}(\xi_{t,\mathrm{x}(t)},\Gamma_{jk1})))\\ &= f^S(u_{jk}) - 2i\frac{\tau_{jk}\, i_{jk}}{4}t\log t\,\Phi^S(u_{jk})\Phi^C(i_{jk})\begin{pmatrix} 0 & -2\\ 2 & 0\end{pmatrix}\Phi^C(i_{jk})^{-1}\Phi^S(u_{jk})^{-1}\mid_{\lambda=1} + O(t)\\ &= f^S(u_{jk}) - i\tau_{jk}t\log t\,\Phi^S(u_{jk})\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}\Phi^S(u_{jk})^{-1}\mid_{\lambda=1} + O(t)\\ &= f^S(u_{jk}) + \tau_{jk}t\log t\, N^S(u_{jk}) + O(t)\end{aligned}$$

and Point (2) follows. □

10.5. **Transition annuli.** For $(j,k)\in E$ and $t>0$, let $\mathcal{A}_{jk,t}$ be the annulus $|t_{jk}|/\varepsilon < |z_{jk,t}| < \varepsilon$ which is identified with the annulus $|t_{jk}|/\varepsilon < |z'_{jk,t}| < \varepsilon$ when opening nodes. We have for $|z_{jk,t}|\leq 1$

(70) $$\frac{1}{2} \leq \frac{|z_{jk,t}|}{|z-p_{jk,t}|} \leq \frac{3}{2}.$$

So provided $|p_{jk,t}-u_{jk}|\leq\frac{\varepsilon}{6}$, which is true for t small enough, the outer boundary component of $\mathcal{A}_{jk,t}$ (namely the circle $|z_{jk,t}|=\varepsilon$) is included in Ω_{j,ε_1}. Likewise, the inner boundary component of $\mathcal{A}_{jk,t}$ (namely the circle $|z'_{jk,t}|=\varepsilon$) is included in $\Omega_{jk,\varepsilon_1}$.

Proposition 22. *For $t>0$ small enough and $(j,k)\in E^+$:*

1. *The images of $\mathcal{A}_{jk,t}$ and $\mathcal{A}_{kj,t}$ by $\widetilde{f}_t$ are graphs over annuli in the plane orthogonal to u_{jk}.*
2. *If $\tau_{jk}>0$, the image of the annulus $\mathcal{A}_{jk,t}\cup\Omega_{jk,\varepsilon_1}\cup\mathcal{A}_{kj,t}$ is embedded.*

Proof:

(1) We may think of the universal covering $\widetilde{\mathcal{A}}_{jk,t}$ of $\mathcal{A}_{jk,t}$ as the Riemann surface on which $\log z_{jk,t}$ is well defined. Let $c>0$ such that for $u\in\mathbb{S}^1$ and $z\in D(u,\frac{1}{2})$

$$\|\Phi^S(z)-\Phi^S(u)\|\leq c|z-u| \quad\text{and}\quad \|\Phi^C(z)-\Phi^C(u)\|\leq c|z-u|.$$

Then for t small enough we have, by Equations (58), (61) and (70)

(71) $$\|\Phi_t - \Phi_0(\widetilde{1}_j)\Phi^S(u_{jk})\| \leq 4c\varepsilon \quad \text{on } \partial\mathcal{A}_{jk,t}$$

We would like to apply the maximum principle to conclude that the same estimate holds inside $\mathcal{A}_{jk,t}$. This is of course not possible because Φ_t is not well defined on $\mathcal{A}_{jk,t}$, but this problem is easily solved as follows. Define on $\widetilde{A}_{jk,t}$

$$G_t = \exp\left(-\frac{\log z_{jk,t}}{2\pi i}\log \mathcal{M}(\Phi_t,\gamma_{jk})\right).$$

Then $G_t\Phi_t$ descends to a well defined holomorphic function on $\mathcal{A}_{jk,t}$. Also, we have $G_t = I_2+O(t)$, so by Equation (71), for t small enough

$$\|G_t\Phi_t - \Phi_0(\widetilde{1}_j)\Phi^S(u_{jk})\| \leq 5c\varepsilon \quad \text{on } \partial\mathcal{A}_{jk,t}.$$

By the maximum principle

$$\|G_t\Phi_t - \Phi_0(\widetilde{1}_j)\Phi^S(u_{jk})\| \leq 5c\varepsilon \quad \text{in } \mathcal{A}_{jk,t}.$$

(The maximum principle for Banach valued holomorphic functions states that if $\|f\|$ has an interior maximum then $\|f\|$ is constant, and is an easy consequence of the Gauss mean value formula.) Hence for t small enough

$$\|\Phi_t - \Phi_0(\widetilde{1}_j)\Phi^S(u_{jk})\| \leq 6c\varepsilon \quad \text{in } \mathcal{A}_{jk,t}.$$

Fix a positive $\alpha < 1/4$. Using that Iwasawa decomposition is differentiable, we have, provided ε is chosen small enough (observe that c is a universal constant)

$$\|\mathrm{Uni}(\Phi_t) - \Phi_0(\widetilde{1}_j)\Phi^S(u_{jk})\| \leq \alpha \quad \text{in } \mathcal{A}_{jk,t}.$$

Let N_t be the Gauss map of f_t. By Equation (5), we obtain since $\Phi_0(\widetilde{1}_j)\mid_{\lambda=1}= I_2$

$$\|N_t - N^S(u_{jk})\| \leq 2\alpha \quad \text{in } \mathcal{A}_{jk,t}$$

Recall that $\vec{\Psi}\circ N^S(u_{jk}) = -u_{jk}$ and let $\pi^\perp_{u_{jk}}$ be the projection on the plane orthogonal to u_{jk}. Then $\pi^\perp_{u_{jk}}\circ\widetilde{f}_t$ is a local diffeomorphism on $\mathcal{A}_{jk,t}$. By Proposition 18, the image by $\pi^\perp_{u_{jk}}\circ\widetilde{f}_t$ of the outer boundary component $|z_{jk,t}| = \varepsilon$ is close to a circle of center $\pi^\perp_{u_{jk}}(v_j)$ and radius of order ε_1. By Proposition 20, the image by $\pi^\perp_{u_{jk}}\circ\widetilde{f}_t$ of the inner boundary component $|z'_{jk,t}| = \varepsilon$ is close to a circle of center $\pi^\perp_{u_{jk}}(v_j)$ and radius of order t. Hence the projection of the inner boundary component is inside the projection of the outer boundary component, so $\pi^\perp_{u_{jk}}\circ\widetilde{f}_t$ is a diffeomorphism onto its image by a standard covering argument. This proves Point (1).

(2) Introduce a coordinate system (x,y,z) with origin $\widetilde{f}_t(i_{jk})$ and x-axis parallel to the line $(v_{j,t},v_{k,t})$. In the following, left and right refer to the x-axis (so $v_{j,t}$ is on the left of $v_{k,t}$). Let S_t be the hemisphere $-1\leq x\leq 0$ of the unit sphere centered at $(-1,0,0)$. Assume $\tau_{jk}>0$. By Proposition 20, the right boundary of $\widetilde{f}_t(\mathcal{A}_{jk,t})$ is on the left of S_t. By Proposition 18, the left boundary of $\widetilde{f}_t(\mathcal{A}_{jk,t})$ is at distance $O(t)$ from the radius 1-sphere centered at $v_{j,t}$ so is on the left of S_t by Point (2) of Proposition 21. Moreover, the mean curvature vector on $\widetilde{f}_t(\mathcal{A}_{jk,t})$ points to the left (because it does so on the left boundary). By the maximum principle, $\widetilde{f}_t(\mathcal{A}_{jk,t})$ is on the left of S_t so in particular lies in the half-space $x<0$. By the same argument, $\widetilde{f}_t(\mathcal{A}_{kj,t})$ lies in the half-space $x>0$. Hence they are disjoint and it is now clear, from Proposition 20 and Point (1), that the image of $\mathcal{A}_{jk,t}\cup\Omega_{jk,\varepsilon_1}\cup\mathcal{A}_{kj,t}$ is embedded. □

10.6. **Embeddedness.** Let M_t be the image of $\widetilde{f}_t$.

Proposition 23. *If all weights τ_{jk} are positive, then for $t>0$ small enough, M_t is Alexandrov-embedded. If moreover the graph Γ is pre-embedded, then for $t>0$ small enough, M_t is embedded.*

Proof: we follow closely the proof of Proposition 7 in [29]. Assume that all τ_{jk} are positive. By Proposition 18 and taking $\varepsilon_1 > 0$ small enough, we may find, for $(j,k) \in E \cup R$, a Jordan curve $\gamma'_{jk,t}$, freely homotopic to γ_{jk}, whose image is in a plane $\Pi_{jk,t}$ orthogonal to $(v_{j,t}, v_{k,t})$, and moreover:

- If $(j,k) \in E$, $\gamma'_{jk,t}$ lies in $\Omega_{j,\varepsilon_1} \cap \mathcal{A}_{jk,t}$,
- If $(j,k) \in R$, $\gamma'_{jk,t}$ lies in $\Omega_{j,\varepsilon_1} \cap D^*(p^0_{jk,t}, \varepsilon_2)$.

Let $\Delta_{jk,t} \subset \Pi_{jk,t}$ be the flat disk bounded by $\widetilde{f}_t(\gamma'_{jk,t})$.

- For $j \in J$, let $\Omega'_{j,t} \subset \Omega_{j,\varepsilon_1}$ be the domain bounded by the curves $\gamma'_{jk,t}$ for $k \in E_j \cup R_j$. By Proposition 18, $\widetilde{f}_t(\Omega'_{j,t})$ is embedded and does not intersect the disks $\Delta_{jk,t}$ for $k \in E_j \cup R_j$. Hence the union of $\widetilde{f}_t(\Omega'_{j,t})$ and $\overline{\Delta}_{jk,t}$ for $k \in E_j \cup R_j$ is the image of a continuous injection of the 2-sphere. By the Jordan Brouwer Theorem, it is the boundary of a bounded domain $W_{j,t}$.
- For $(j,k) \in R$, let $D_{jk,t}$ be the disk bounded by $\gamma'_{jk,t}$ and $D^*_{jk,t} = D_{jk,t} \setminus \{p^0_{jk,t}\}$. By Proposition 19, $\widetilde{f}_t(D^*_{jk,t})$ is embedded. By the proof of Claim 3 in [29], its reunion with $\overline{\Delta}_{jk,t}$ bounds a cylindrically bounded domain $W_{jk,t}$.
- For $(j,k) \in E^+$, let $\mathcal{A}'_{jk,t} \subset \mathcal{A}_{jk,t} \cup \Omega_{jk,\varepsilon_1} \cup \mathcal{A}_{kj,t}$ be the annulus bounded by $\gamma'_{jk,t}$ and $\gamma'_{kj,t}$. By Proposition 22, $\widetilde{f}_t(\mathcal{A}'_{jk,t})$ is embedded. By Claim 1 below and the Jordan Brouwer Theorem, $\widetilde{f}_t(\mathcal{A}'_{jk,t}) \cup \overline{\Delta}_{jk,t} \cup \overline{\Delta}_{kj,t}$ is the boundary of a bounded domain $W_{jk,t}$.

Let W_t be the closed manifold with boundary obtained as the disjoint union of all $\overline{W}_{j,t}$ for $j \in J$ and $\overline{W}_{jk,t}$ for $(j,k) \in E \cup R$, identifying $\overline{W}_{j,t}$ and $\overline{W}_{jk,t}$ for $k \in E_j \cup R_j$ along their common boundary $\Delta_{jk,t}$. Let $F_t : W_t \to \mathbb{R}^3$ be the canonical injection on each $\overline{W}_{j,t}$ and $\overline{W}_{jk,t}$. Note that F_t is a priori not injective, since the domains may overlap (its image $F_t(W_t)$ is what is called an immersed domain.) But F_t is a proper local diffeomorphism whose boundary restriction parametrizes M_t. Moreover, we may compactify W_t by adding one point per domain $\overline{W}_{jk,t}$ for $(j,k) \in R$. This proves that M_t is Alexandrov embedded.

Assume now that Γ is pre-embedded. Then the domains $W_{j,t}$ for $j \in J$ and $W_{jk,t}$ for $(j,k) \in E \cup R$ are disjoint, and their closures intersect only along the disks $\Delta_{jk,t}$. Hence the map F_t is an embedding so M_t is embedded. □

Claim 1. *We may choose the curves $\gamma'_{jk,t}$ and $\gamma'_{kj,t}$ so that $\widetilde{f}_t(\mathcal{A}'_{jk,t})$ does not intersect the disks $\Delta_{jk,t}$ and $\Delta_{kj,t}$.*

Proof: we continue with the coordinate system (x, y, z) introduced in the proof of Point (2) of Proposition 22. By Proposition 18, we may find a Jordan curve $\gamma''_{jk,t}$ in $\Omega_{j,\varepsilon_1} \cap \mathcal{A}_{jk,t}$ whose image is at constant distance from the x-axis. Let $\mathcal{A}''_{jk,t}$ be the annulus bounded by $\gamma''_{jk,t} \cup \gamma'_{kj,t}$ and $A''_t = \widetilde{f}_t(\mathcal{A}''_{jk,t})$. Consider half a period of a Delaunay surface D_t with axis Ox and necksize $\tau_{jk}t/2$, bounded on the left by a circle of maximum radius and on the right by a circle of radius $\tau_{jk}t/2$. Translate the Delaunay surface D_t from the left until a first contact point p''_t with A''_t occurs. By Propositions 20 and 22, p''_t cannot be on the right boundary of D_t (which is too small) nor on the left boundary of D_t (which is too big). By the maximum principle, p''_t must be on the left boundary of A''_t and has minimum x-coordinate. Choose the curve $\gamma'_{jk,t}$ so that $\Pi_{jk,t}$ is the plane orthogonal to the x-axis and containing p''_t. Then A''_t, being on the right of $\Pi_{jk,t}$, does not intersect $\Delta_{jk,t}$. The annulus bounded by $\gamma'_{jk,t}$ and $\gamma''_{jk,t}$ is inside Ω_{j,ε_1} so its image does not intersect $\Delta_{jk,t}$ by Proposition 18. Hence $\widetilde{f}(\mathcal{A}'_{jk,t})$ does not intersect $\Delta_{jk,t}$, and in the same way, it does not intersect $\Delta_{kj,t}$. □

This concludes the proof of Theorem 1.

Appendix A. A regularity result

In this section we prove a regularity result in the spirit of Theorem 5 in [30] or Theorem 6 in [12]. The philosophy of these results is to idenfity which part of the Regularity Problem is solved when the Monodromy Problem around a singularity is solved. For use in other papers, we consider the Monodromy Problem associated to the general Sym-Bobenko formula in space forms with Sym-points at λ_1, λ_2, with either

(1) $\lambda_1 = \lambda_2 = 1$ ($\mathbb{R}^3$ case)
(2) $\lambda_1 = e^{i\theta}, \lambda_2 = e^{-i\theta}$ with $0 < \theta < \pi$ ($\mathbb{S}^3$ case)
(3) $\lambda_1 = e^q$, $\lambda_2 = e^{-q}$ with $q > 0$ ($\mathbb{H}^3$ case).

The Monodromy Problem in cases (2) and (3) is

$$\left\{\begin{array}{l}\mathcal{M}(\Phi,\gamma) \in \Lambda SU(2)\\ \mathcal{M}(\Phi,\gamma)\mid_{\lambda_1} = \mathcal{M}(\Phi,\gamma)\mid_{\lambda_2} = \pm I_2\end{array}\right. \tag{72}$$

Theorem 4. *Let $\Omega \subset \overline{\mathbb{C}}$ be a σ-symmetric domain containing the points 0, ∞ and not containing -1. Let $\xi_t = \begin{pmatrix}\alpha_t & \lambda^{-1}\beta_t\\ \gamma_t & -\alpha_t\end{pmatrix}$ be a C^1 family of σ-symmetric DPW potentials with the following properties:*

(1) *α_t, β_t are holomorphic in Ω and γ_t has at most a double pole at 0 and ∞,*
(2) $\text{Re}(\text{Res}_0(z\gamma_t)) = 0$,
(3) $\text{Res}_0(\gamma_t^0) = 0$.

Assume that there exists a continuous family of σ-symmetric solutions Φ_t of $d\Phi_t = \Phi_t\xi_t$ in the universal covering of $\Omega \setminus \{0,\infty\}$ and a σ-symmetric curve $\delta \subset \Omega$ bounding a disk-type domain containing 0 and ∞, such that the Monodromy Problem (7) or (72) for $\mathcal{M}(\Phi_t,\delta)$ is solved. Further assume that at $t = 0$

$$\xi_0 = \begin{pmatrix}0 & \lambda^{-1}\\ 0 & 0\end{pmatrix}\frac{k\,dz}{(z+1)^2}$$

with $k \in \mathbb{R}^*$ and $\Phi_0(1)$ is diagonal. Then for t in a neighborhood of 0, ξ_t is holomorphic at 0 and ∞.

Proof: Let (F,B) be the Iwasawa decomposition of $\Phi_0(1)$ (both factors are diagonal). Replacing Φ_t by $F^{-1}\Phi_t$, we may assume that $\Phi_0(1)$ is a diagonal matrix in $\Lambda_{\mathbb{R}}^+ SL(2,\mathbb{C})$ so

$$\Phi_0(z) = \begin{pmatrix}\rho & 0\\ 0 & \frac{1}{\rho}\end{pmatrix}\begin{pmatrix}1 & \frac{k(z-1)}{2\lambda(z+1)}\\ 0 & 1\end{pmatrix}$$

with $\rho \in \mathcal{W}_{\mathbb{R}}^{\geq 0}$. By Hypothesis (2) and (3), we may write

$$\text{Res}_0(z\gamma_t) = ia_t \quad \text{and} \quad \text{Res}_0(\gamma_t) = \lambda(b_t + ic_t)$$

with $a_t, b_t, c_t \in \mathcal{W}_{\mathbb{R}}^{\geq 0}$. For $\mathrm{x} = (a,b,c) \in (\mathcal{W}_{\mathbb{R}}^{\geq 0})^3$, define

$$\omega_{\mathrm{x}} = ia(1-z^2)\frac{dz}{z^2} + \lambda\big(b + ic\frac{1-z}{1+z}\big)\frac{dz}{z}.$$

Then

$$\overline{\sigma^*\omega_{\mathrm{x}}} = -\omega_{\mathrm{x}}, \quad \text{Res}_0(z\omega_{\mathrm{x}}) = ia \quad \text{and} \quad \text{Res}_0\omega_x = \lambda(b+ic).$$

Writing $\mathrm{x}_t = (a_t, b_t, c_t)$, we see that $\gamma_t - \omega_{\mathrm{x}_t}$ is holomorphic at 0 and ∞ by symmetry. Define

$$\xi_{t,\mathrm{x}} = \begin{pmatrix}\alpha_t & \lambda^{-1}\beta_t\\ \gamma_t - \omega_{\mathrm{x}_t} & -\alpha_t\end{pmatrix} + \begin{pmatrix}0 & 0\\ \omega_{\mathrm{x}} & 0\end{pmatrix}.$$

(The first term is holomorphic in Ω). Let $\Phi_{t,\mathrm{x}}$ be the solution of $d\Phi_{t,\mathrm{x}} = \Phi_{t,\mathrm{x}}\xi_{t,\mathrm{x}}$ with initial condition $\Phi_{t,\mathrm{x}}(z_0) = \Phi_t(z_0)$, where z_0 is an arbitrary base point. Then $\xi_{t,\mathrm{x}_t} = \xi_t$ and $\Phi_{t,\mathrm{x}_t} = \Phi_t$. Since ξ_0 is holomorphic at 0, we have $\mathrm{x}_0 = 0$. Theorem 4 follows from the following

Lemma 1. *For (t, x) in a neighborhood of $(0,0)$, the only solution to the Monodromy Problem (7) or (72) for $\mathcal{M}(\Phi_{t,\mathrm{x}},\delta)$ is $\mathrm{x}=0$.*

Proof: let

$$M(t,\mathrm{x}) = H\log\mathcal{M}(\Phi_{t,\mathrm{x}},\delta)H^{-1} \quad \text{with} \quad H=\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}\in\Lambda SU(2).$$

(The reason to conjugate by H will be clear in a moment.) By Proposition 8 in [28], the partial differential of M with respect to x at $(0,0)$, applied to the vector $\mathrm{x}=(a,b,c)$, is given by

$$d_{\mathrm{x}}M(0,0)\cdot\mathrm{x} = \int_\delta N\omega_{\mathrm{x}}$$

where

$$N = H\Phi_0\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\Phi_0^{-1}H^{-1} = \begin{pmatrix}\frac{k(z-1)}{2\lambda(z+1)} & \frac{-k^2\rho^2(z-1)^2}{4\lambda(z+1)^2}\\ \frac{1}{\lambda\rho^2} & -\frac{k(z-1)}{2\lambda(z+1)}\end{pmatrix}.$$

Since $N\omega_x$ has only poles at 0, -1 and ∞, we have by the Residue Theorem

$$d_{\mathrm{x}}M(0,0)\cdot\mathrm{x} = 2\pi i\big(\mathrm{Res}_0(N\omega_{\mathrm{x}})+\mathrm{Res}_\infty(N\omega_{\mathrm{x}})\big) = -2\pi i\,\mathrm{Res}_{-1}(N\omega_{\mathrm{x}}).$$

Computing the residue at $z=-1$, we obtain

$$d_{\mathrm{x}}M(0,0) = 2\pi i\begin{pmatrix}-k\,db & \lambda^{-1}ik^2\rho^2(2da-\lambda dc/2)\\ 2i\,dc/\rho^2 & k\,db\end{pmatrix}.$$

Define

$$\widetilde{a}=-kb, \quad \widetilde{b}=k^2\rho^2(2a-\lambda c/2) \quad \text{and} \quad \widetilde{c}=2c/\rho^2.$$

It is clear that $(a,b,c)\mapsto(\widetilde{a},\widetilde{b},\widetilde{c})$ is an automorphism of $(\mathcal{W}_{\mathbb{R}}^{\geq 0})^3$. The point of this change of variables (and the conjugation by H) is that we now have

$$d_{\mathrm{x}}M(0,0) = 2\pi i\begin{pmatrix}d\widetilde{a} & \lambda^{-1}i\,d\widetilde{b}\\ i\,d\widetilde{c} & -d\widetilde{a}\end{pmatrix}.$$

This is precisely Equation (51) with a_{jk}, b_{jk}, c_{jk} replaced by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ and $r_{jk}=0$. So in the $\mathbb{R}^3$ case, the proof of Point (3) of Proposition 11 yields that for t in a neighborhood of 0, the Monodromy Problem (7) uniquely determines $(\widetilde{a},\widetilde{b},\widetilde{c})$, hence x, as a function of t. Now $\mathrm{x}=0$ is a trivial solution (since $\xi_{t,0}$ is holomorphic in Ω) so $\mathrm{x}=0$ is the unique solution. In the $\mathbb{S}^3$ case (respectively the $\mathbb{H}^3$ case), the Monodromy Problem is equivalent, using the ρ-symmetry, to

$$\begin{cases}M\in\Lambda\mathfrak{su}(2)\\ \mathrm{Im}(M_{11}\mid_{\lambda=e^{i\theta}})=0\\ M_{21}\mid_{\lambda=e^{i\theta}}=0\end{cases} \quad \text{respectively} \quad \begin{cases}M\in\Lambda\mathfrak{su}(2)\\ \mathrm{Im}(M_{11}\mid_{\lambda=e^{q}})=0\\ \mathrm{Re}(M_{12}\mid_{\lambda=e^{q}})=0\\ \mathrm{Re}(M_{21}\mid_{\lambda=e^{q}})=0\end{cases}$$

In the $\mathbb{S}^3$ case, define $\mathcal{F}$, $\mathcal{G}$ by Equations (48) and (49) with M in place of $\breve{M}_{jk}$ and

$$\mathcal{E}=(\mathcal{E}_i)_{1\leq i\leq 5} = \big(\mathcal{F}^+,\mathcal{G}^+,\lambda(\mathcal{G}^-)^*,\mathrm{Im}(M_{11}\mid_{\lambda=e^{i\theta}}),M_{21}\mid_{\lambda=e^{i\theta}}\big).$$

Then

$$d\mathcal{E}_1 = -2\pi d\widetilde{a}^+$$
$$d\mathcal{E}_2 = -2\pi(d\widetilde{b}^+ + \lambda d\widetilde{c}^0)$$
$$d\mathcal{E}_3 = -2\pi(d\widetilde{c}^+ + \lambda d\widetilde{b}^0)$$
$$d\mathcal{E}_4 + \mathrm{Re}(d\mathcal{E}_1\mid_{\lambda=e^{i\theta}}) = 2\pi d\widetilde{a}^0$$
$$d\mathcal{E}_5 - d\mathcal{E}_3\mid_{\lambda=e^{i\theta}} = 2\pi(e^{i\theta}d\widetilde{b}^0 - d\widetilde{c}^0).$$

It easily follows (since $e^{i\theta} \notin \mathbb{R}$) that $d\mathcal{E}(0,0)$ is an isomorphism from $\mathcal{W}_{\mathbb{R}}^{>0} \times \mathbb{R}^3$ to $\mathcal{W}_{\mathbb{R}}^{>0} \times \mathbb{R} \times \mathbb{C}$. Again, we conclude with the Implicit Function Theorem that the only solution of the Monodromy Problem is $\mathrm{x} = 0$. We omit the proof in the $\mathbb{H}^3$ case which is similar. □

By duality, we obtain the following result (with the same hypothesis on Ω and δ):

Corollary 1. *Let ξ_t be a C^1 family of σ-symmetric DPW potentials on Ω with the following properties:*

1. *α_t, γ_t are holomorphic in Ω and β_t has at most a double pole at 0 and ∞,*
2. $\mathrm{Re}(\mathrm{Res}_0(z\beta_t)) = 0$,
3. $\mathrm{Res}_0(\beta_t^0) = 0$.

Assume that there exists a continuous family of σ-symmetric solutions Φ_t of $d\Phi_t = \Phi_t\xi_t$ such that the Monodromy Problem (7) or (72) for $\mathcal{M}(\Phi_t, \delta)$ is solved. Further assume that at $t = 0$

$$\xi_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{k\,dz}{(z+1)^2}$$

and $\Phi_0(1)$ is diagonal. Then for t in a neighborhood of 0, ξ_t is holomorphic at 0 and ∞.

Appendix B. Principal solution through a neck

Fix some numbers $0 < \varepsilon' < \varepsilon$. For $t \in \mathbb{C}$ such that $0 < |t| < \varepsilon^2$, let $\mathcal{A}_t \subset \mathbb{C}$ be the annulus $|t|/\varepsilon < |z| < \varepsilon$ and $\psi_t : \mathcal{A}_t \to \mathcal{A}_t$ be the involution defined by $\psi_t(z) = t/z$. We see an element of the universal cover $\widetilde{\mathbb{C}^*}$ of $\mathbb{C}^*$ as a complex number $t \in \mathbb{C}^*$ with a determination of its argument (which we do not write), so the function $\log t$ is well defined on $\widetilde{\mathbb{C}^*}$. We denote $t \mapsto e^{2\pi i}t$ the Deck transformation of $\widetilde{\mathbb{C}^*}$ which increases the argument of t by 2π. For $t \in \widetilde{\mathbb{C}^*}$, let β_t be the curve from ε' to t/ε' parametrized for $s \in [0,1]$ by

$$\beta_t(s) = (\varepsilon')^{1-2s}t^s = (\varepsilon')^{1-2s}e^{s\log t}.$$

Our goal is to understand the limit behavior of $\mathcal{P}(\xi_t, \beta_t)$ as $t \to 0$, under suitable hypothesis on the potential ξ_t. Let γ be the circle parametrized by $\gamma(s) = \varepsilon' e^{2\pi i s}$.

Theorem 5. *Let ξ_t be a family of $\Lambda\mathfrak{sl}(n,\mathbb{C})$ valued holomorphic 1-forms on $\mathcal{A}_t$, depending holomorphically on $t \in D^*(0, \varepsilon^2)$, and let $\widehat{\xi}_t = \psi_t^*\xi_t$. Assume that*

$$\lim_{t\to 0} \xi_t = \xi_0 \qquad \textit{and} \qquad \lim_{t\to 0} \widehat{\xi}_t = \widehat{\xi}_0$$

where ξ_0 and $\widehat{\xi}_0$ are holomorphic in $D(0,\varepsilon)$ and the limit is uniform on compact subsets of $D^*(0,\varepsilon)$. Define for $t \in \widetilde{\mathbb{C}^*}$ small enough

$$\widetilde{F}(t) = \mathcal{P}(\xi_t, \gamma)^{-\frac{\log t}{2\pi i}}\mathcal{P}(\xi_t, \beta_t).$$

Then

1. *The function $\widetilde{F}$ satisfies $\widetilde{F}(e^{2\pi i}t) = \widetilde{F}(t)$ so descends to a well defined holomorphic function $F(t)$ defined in a punctured neighborhood of 0.*
2. *The function F extends holomorphically at $t = 0$ with*
$$F(0) = \mathcal{P}(\xi_0, \varepsilon', 0)\mathcal{P}(\widehat{\xi}_0, 0, \varepsilon').$$
3. *If $t > 0$, the function $\mathcal{P}(\xi_t, \beta_t)$ extends to a smooth function of t and $t\log t$ with value $F(0)$ at $t = 0$. Moreover we have as $t \to 0$*
$$\mathcal{P}(\xi_t, \beta_t) = \left(I_2 + \frac{t\log t}{2\pi i}\frac{\partial}{\partial t}\mathcal{P}(\xi_t, \gamma)\mid_{t=0}\right)F(0) + O(t).$$

Remark 14. We apply Theorem 5 in the proof of Proposition 13 with $\xi_t = (z_{jk}^{-1})^*\xi_{t,\mathrm{x}}$ and $t = t_{jk}$. Then $z'_{jk} = \psi_t \circ z_{jk}$ so $\widehat{\xi}_t = ((z'_{jk})^{-1})^*\xi_{t,\mathrm{x}}$. By Proposition 3, ξ_0 and $\widehat{\xi}_0$ are both holomorphic in $D(0,\varepsilon)$, with $\xi_0 = (z_{jk}^{-1})^*(M_j\omega_0)$ and $\widehat{\xi}_0 = ((z'_{jk})^{-1})^*(M_{jk}\omega_{q_{jk}})$. Theorem 5 says that $\mathcal{P}((z_{jk}^{-1})^*\xi_{t,\mathrm{x}}, \varepsilon', t_{jk}/\varepsilon')$ extends at $t = 0$ to a smooth function of t_{jk} and $t_{jk}\log t_{jk}$ with value at $t_{jk} = 0$

$$\mathcal{P}((z_{jk}^{-1})^*(M_j\omega_0), \varepsilon', 0)\,\mathcal{P}(((z'_{jk})^{-1})^*(M_{jk}\omega_{q_{jk}}), 0, \varepsilon').$$

To justify that the extension is a smooth function of $t, t\log t$ and x we use Hartog Theorem on separate holomorphy to ensure that the function F depends holomorphically on (t,x). In other words, $\mathcal{P}(\xi_{t,\mathrm{x}}, z_{jk} = \varepsilon', z'_{jk} = \varepsilon')$ extends to a smooth function of t, $t\log t$ and x with value at $t = 0$

$$\mathcal{P}(M_j\omega_0, z_{jk} = \varepsilon', z_{jk} = 0)\,\mathcal{P}(M_{jk}\omega_{q_{jk}}, z'_{jk} = 0, z'_{jk} = \varepsilon') = \mathcal{P}(\xi_{0,\mathrm{x}}, z_{jk} = \varepsilon', z'_{jk} = \varepsilon')$$

where in the last expression, it is understood that the principal solution is continuous at the node. This gives some theoretical ground for the heuristic explained in Section 4.

Proof: first of all, by the change of variables $z' = z/\varepsilon'$ and $t' = t/(\varepsilon')^2$, we may assume without loss of generality that $\varepsilon' = 1$ (so $\varepsilon > 1$). The expression of β_t simplifies to $\beta_t(s) = t^s$. The restriction of ξ_t to the unit circle γ extends holomorphically at $t = 0$, with value ξ_0. Since ξ_0 is holomorphic in $D(0,\varepsilon)$, $\mathcal{P}(\xi_0,\gamma) = I_2$. Hence $\log\mathcal{P}(\xi_t,\gamma)$ and $\widetilde{F}(t)$ are well defined for t small enough. Point (1) follows from the fact that the path $\beta_{e^{2\pi i}t}$ is homotopic to $\gamma\beta_t$. To prove Point (2), we split the path β_t into $\beta_t = \alpha_t\widehat{\alpha}_t^{-1}$ where

$$\alpha_t(s) = \beta_t(s/2) = t^{s/2} \quad\text{ and }\quad \widehat{\alpha}_t(s) = \beta_t(1-s/2) = \psi_t(\alpha_t(s)).$$

Since $F(t)$ is well-defined, we may assume that $|\arg t| \le \pi$. Provided $|t| \le e^{-\pi}$, we have $|\log t| \le 2|\log|t||$ so

$$|\alpha'_t(s)| = \tfrac{1}{2}|t|^{s/2}|\log t| \le |t|^{s/2}|\log|t||. \tag{73}$$

Integrating the estimate (73), we see that the length of the spiral α_t is bounded by 2.

Lemma 2. *There exists a uniform constant C such that for t small enough enough:*

$$\int_0^1 \|\left[\xi_t(\alpha_t(s)) - \xi_0(\alpha_t(s))\right]\alpha'_t(s)\|\,ds \le C|t|^{1/2}. \tag{74}$$

Proof: we use the letter C to denote various uniform constants. Fix some $\varepsilon_2 \in (1,\varepsilon)$. On the circle $C(0,\varepsilon_2)$, ξ_t depends holomorphically on t in a neighborhood of 0 so

$$\int_{C(0,\varepsilon_2)} \|\xi_t - \xi_0\| \le C|t|. \tag{75}$$

By the change of variable formula, the convergence of $\widehat{\xi}_t$ to $\widehat{\xi}_0$ and the holomorphicity of ξ_0 and $\widehat{\xi}_0$ in $D(0,\varepsilon)$:

$$\int_{C(0,|t|/\varepsilon_2)} \|\xi_t - \xi_0\| \le \int_{C(0,\varepsilon_2)} \|\widehat{\xi}_t\| + \int_{C(0,|t|/\varepsilon_2)} \|\xi_0\| \le C. \tag{76}$$

We expand $\xi_t - \xi_0$ in Laurent series in the annulus $|t|/\varepsilon_2 \le |z| \le \varepsilon_2$ as

$$\xi_t(z) - \xi_0(z) = \sum_{k\in\mathbb{Z}} A_k(t) z^k dz$$

where the matrices $A_k(t)$ are given by

$$A_k(t) = \frac{1}{2\pi i}\int_{C(0,\varepsilon_2)} \frac{\xi_t(z) - \xi_0(z)}{z^{k+1}} = \frac{1}{2\pi i}\int_{C(0,|t|/\varepsilon_2)} \frac{\xi_t(z) - \xi_0(z)}{z^{k+1}}.$$

Estimates (75) and (76) give us respectively:

$$\|A_k(t)\| \leq C\frac{|t|}{\varepsilon_2^{k+1}} \quad \text{and} \quad \|A_k(t)\| \leq C\frac{\varepsilon_2^{k+1}}{|t|^{k+1}}. \tag{77}$$

Then we have the following estimates:

$$\begin{aligned}
\int_0^1 \|[\xi_t(\alpha_t(s)) - \xi_0(\alpha_t(s))]\alpha_t'(s)\| \, ds &\leq \sum_{k\in\mathbb{Z}} \int_0^1 \|A_k(t)\| \, |\alpha_t(s)|^k \, |\alpha_t'(s)| \, ds \\
\leq & \quad \sum_{k\in\mathbb{Z}} \int_0^1 \|A_k(t)\| \, |t|^{(k+1)s/2} \, |\log|t|\,| \, ds \quad \text{using (73)} \\
= & \quad \|A_{-1}(t)\| \, |\log|t|\,| + \sum_{k\neq -1} \frac{2}{k+1}\|A_k(t)\| \left(1 - |t|^{(k+1)/2}\right) \\
\leq & \quad \|A_{-1}(t)\| \, |\log|t|\,| + 2\sum_{k\geq 0} \|A_k(t)\| + 2\sum_{k\leq -2} \|A_k(t)\| \, |t|^{\frac{k+1}{2}} \\
\leq & \quad C|t\log|t|\,| + C\sum_{k\geq 0} \frac{|t|}{\varepsilon_2^{k+1}} + C\sum_{k\leq -2} \left(\frac{\varepsilon_2}{|t|^{1/2}}\right)^{k+1} \quad \text{using (77)} \\
\leq & \quad C|t\log|t|\,| + C|t| + C|t|^{\frac{1}{2}}.
\end{aligned}$$

□

Returning to the proof of Theorem 5, let Φ_0 be the solution of $d\Phi_0 = \Phi_0\xi_0$ in $D(0,\varepsilon)$ with initial condition $\Phi_0(1) = I_n$. Let $Y_t(s)$ be the solution on $[0,1]$ of the Cauchy Problem

$$\begin{cases} Y_t'(s) = Y_t(s)\xi_t(\alpha_t(s))\alpha_t'(s) \\ Y_t(0) = I_2. \end{cases}$$

By definition, $\mathcal{P}(\xi_t, \alpha_t) = Y_t(1)$. Define

$$Z_t(s) = Y_t(s) - \Phi_0(\alpha_t(s)).$$

Then

$$\begin{aligned}
Z_t'(s) &= Y_t(s)\xi_t(\alpha_t(s))\alpha_t'(s) - \Phi_0(\alpha_t(s))\xi_0(\alpha_t(s))\alpha_t'(s) \\
&= Z_t(s)\xi_t(\alpha_t(s))\alpha_t'(s) + \Phi_0(\alpha_t(s))[\xi_t(\alpha_t(s)) - \xi_0(\alpha_t(s))]\alpha_t'(s).
\end{aligned}$$

Hence

$$\begin{aligned}
\|Z_t(s)\| = \left\| \int_0^s Z_t'(x)dx \right\| &\leq \int_0^s \|Z_t(x)\| \, \|\xi_t(\alpha_t(x))\alpha_t'(x)\|dx \\
+ & \int_0^s \|\Phi_0(\alpha_t(x))\| \, \| \left[\xi_t(\alpha_t(x)) - \xi_0(\alpha_t(x))\right] \alpha_t'(x)\|dx
\end{aligned}$$

By Grönwall inequality:

$$\|Z_t(1)\| \leq \int_0^1 \|\Phi_0(\alpha_t(s))\| \, \| \left[\xi_t(\alpha_t(s)) - \xi_0(\alpha_t(s))\right] \alpha_t'(s)\|ds \times \exp\left(\int_0^1 \|\xi_t(\alpha_t(s))\alpha_t'(s)\|ds\right).$$

Using Lemma 2, uniform bounds for Φ_0 and ξ_0 in $D(0,1)$ and the length of α_t, we obtain

$$\|\mathcal{P}(\xi_t, \alpha_t) - \Phi_0(\alpha_t(1))\| = \|Z_t(1)\| \leq C|t|^{1/2}.$$

Since Φ_0 is holomorphic in $D(0,1)$,

$$\|\Phi_0(\alpha_t(1)) - \Phi_0(0)\| \leq C|\alpha_t(1)| = C|t|^{1/2}$$

Hence

$$\|\mathcal{P}(\xi_t,\alpha_t)-\Phi_0(0)\|\leq C|t|^{1/2}. \tag{78}$$

Let $\widehat{\Phi}_0$ be the solution of $d\widehat{\Phi}_0=\widehat{\Phi}_0\widehat{\xi}_0$ with initial condition $\widehat{\Phi}_0(1)=I_n$. By the same argument, we have

$$\|\mathcal{P}(\xi_t,\widehat{\alpha}_t)-\widehat{\Phi}_0(0)\|\leq C|t|^{1/2}. \tag{79}$$

By Equations (78) and (79):

$$\|\mathcal{P}(\xi_t,\beta_t)-\Phi_0(0)\widehat{\Phi}_0(0)^{-1}\|=\|\mathcal{P}(\xi_t,\alpha_t)\mathcal{P}(\xi_t,\widehat{\alpha}_t)^{-1}-\Phi_0(0)\widehat{\Phi}_0(0)^{-1}\|\leq C|t|^{1/2}.$$

Since $\mathcal{P}(\xi_t,\gamma)=I_2+O(t)$, we finally obtain

$$\left\|F(t)-\Phi_0(0)\widehat{\Phi}_0(0)^{-1}\right\|\leq C|t|^{1/2}.$$

By Riemann Extension Theorem, F extends holomorphically at $t=0$, and

$$F(0)=\Phi_0(0)\widehat{\Phi}_0(0)^{-1}=\mathcal{P}(\xi_0,1,0)\mathcal{P}(\widehat{\xi}_0,0,1).$$

Finally, to prove Point (3), assume that $t>0$ and write

$$\mathcal{P}(\xi_t,\beta_t)=\exp\left(\frac{t\log t}{2\pi i}t^{-1}\log\mathcal{P}(\xi_t,\gamma)\right)F(t).$$

Since $\mathcal{P}(\xi_0,\gamma)=I_2$, $t^{-1}\log\mathcal{P}(\xi_t,\gamma)$ extends holomorphically at $t=0$ with value $\frac{\partial}{\partial t}\mathcal{P}(\xi_t,\gamma)\mid_{t=0}$ and Point (3) follows. □

Appendix C. Differentiability of smooth functions of t and $t\log t$

Proposition 24. *Let E be a finite dimensional space and $g(t,s,z)$ be a smooth function from a neighborhood of $(0,0,z_0)$ in $\mathbb{R}^2\times E$ to a normed space F. Define*

$$f(t,z)=\begin{cases} g(t,t\log|t|,z) & \text{if } t\neq 0\\ g(0,0,z) & \text{if } t=0.\end{cases}$$

Assume that $g(0,s,z)$ only depends on z. Then f is of class C^1 and

$$df(0,z)=\frac{\partial g}{\partial t}(0,0,z)dt+d_zg(0,0,z).$$

Proof: f is clearly continuous. For $t\neq 0$, we have by the chain rule:

$$df(t,z)=\frac{\partial g}{\partial t}(t,t\log|t|,z)dt+\frac{\partial g}{\partial s}(t,t\log|t|,z)(1+\log|t|)dt+d_zg(t,t\log|t|,z).$$

From the hypothesis, $\frac{\partial g}{\partial s}(0,s,z)=0$ so

$$\|\frac{\partial g}{\partial s}(t,s,z)\|=\|\frac{\partial g}{\partial s}(t,s,z)-\frac{\partial g}{\partial s}(0,s,z)\|=O(t).$$

Hence

$$\lim_{(t,z)\to(0,z_0)}df(t,z)=\frac{\partial g}{\partial t}(0,0,z_0)dt+d_zg(0,0,z_0).$$

It follows (using the Mean Value Inequality) that f is differentiable at $(0,z_0)$ and that it is of class C^1. □

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