GLUING DELAUNAY ENDS TO MINIMAL n-NOIDS USING THE DPW METHOD

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Abstract: we construct constant mean curvature surfaces in euclidean space by gluing n half Delaunay surfaces to a non-degenerate minimal n-noid, using the DPW method.

1. Introduction

In [3], Dorfmeister, Pedit and Wu have shown that surfaces with non-zero constant mean curvature (CMC for short) in euclidean space admit a Weierstrass-type representation, which means that they can be represented in terms of holomorphic data. This representation is now called the DPW method. In [18], we used the DPW method to construct CMC n-noids: genus zero, CMC surfaces with n Delaunay type ends. These n-noids can be described as a unit sphere with n half Delaunay surfaces with small necksizes attached at prescribed points. They had already been constructed by Kapouleas in [11] using PDE methods.

In the case $n = 3$, Alexandrov-embedded CMC trinoids have been classified by Große Brauckman, Kusner and Sullivan in [9]. In particular, equilateral CMC trinoids form a 1-parameter family, parametrized on an open interval. On one end, equilateral trinoids degenerate like the examples described above: they look like a sphere with 3 half Delaunay surfaces with small necksizes attached at the vertices of a spherical equilateral triangle. On the other end, equilateral trinoids limit, after suitable blow-up, to a minimal 3-noid: a genus zero minimal surface with 3 catenoidal ends (see Figure 1).

It seems natural to ask if one can generalize this observation and construct CMC n-noids by gluing half Delaunay surfaces with small necksizes to a minimal n-noid. This is indeed the case, and has been done by Mazzeo and Pacard in [14] using PDE methods. In this paper, we propose a quite simple and natural DPW potential to construct these examples. We prove:

Theorem 1. *Let $n \geq 3$ and let M_0 be a non-degenerate minimal n-noid. There exists a smooth family of CMC surfaces $(M_t)_{0<|t|<\epsilon}$ with the following properties:*

1. *M_t has genus zero and n Delaunay ends.*
2. *$\frac{1}{t}M_t$ converges to M_0 as $t \to 0$.*
3. *If M_0 is Alexandrov-embedded, all ends of M_t are of unduloid type if $t > 0$ and of nodoid type if $t < 0$. Moreover, M_t is Alexandrov-embedded if $t > 0$.*

Non-degeneracy of a minimal n-noid will be defined in Section 2. The two surfaces M_t and M_{-t} are geometrically different: if M_t has an end of unduloid type then the corresponding end of M_{-t} is of nodoid type. See Proposition 6 for more details. Of course, a minimal n-noid is never embedded if $n \geq 3$ so the surfaces M_t are not embedded. Alexandrov-embedded

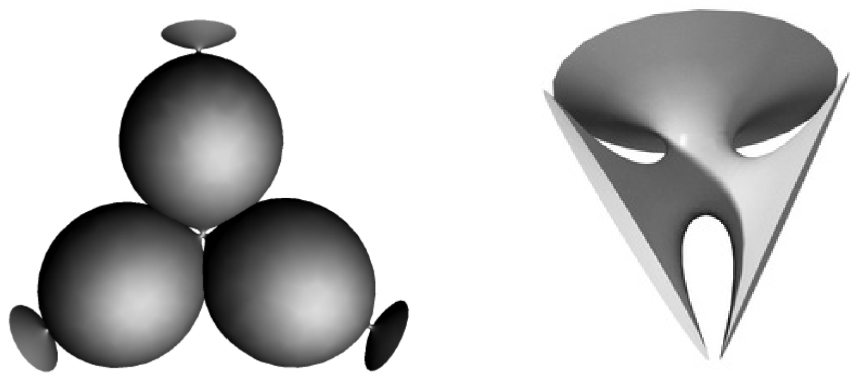

Figure 1. A CMC 3-noid (left, image by N. Schmitt [16]) and a minimal 3-noid (right). There is a tiny "copy" of the minimal 3-noid at the center of the CMC 3-noid.

minimal n-noids whose ends have coplanar axes have been classified by Cosin and Ros in [2], and Alexandrov-embedded CMC n-noids whose ends have coplanar axes have been classified by Große-Brauckmann, Kusner and Sullivan in [10].

As already said, these surfaces have already been constructed in [14]. Our motivation to construct them with the DPW method is to answer the following questions:

(1) How can we produce a DPW potential from the Weierstrass data (g, ω) of the minimal n-noid M_0 ?
(2) How can we prove, with the DPW method, that $\frac{1}{t}M_t$ converges to M_0 ?

The answer to Question 2 is Theorem 4 in Section 4, a general blow-up result in the context of the DPW method. In [19], we use the DPW method to construct higher genus CMC surfaces with small necks. Theorem 4 is used to ensure that the necks have asymptotically catenoidal shape.

Remark 1. *As the referee pointed out, the relation between minimal surfaces and CMC-1 surfaces in the DPW framework has already been investigated by Brander and Dorfmeister* [1]*. In that paper, the authors propose a DPW potential from the Weierstrass data of a minimal surface M. The Monodromy Problem is not addressed, however, so the resulting CMC-1 surfaces do not close, unless M is simply connected.*

I would like to thank the referee for his helpful comments and for providing the references [7] and [8].

2. Non-degenerate minimal n-noids

A minimal n-noid is a complete, immersed minimal surface in $\mathbb{R}^3$ with genus zero and n catenoidal ends. Let M_0 be a minimal n-noid and (Σ, g, ω) its Weierstrass data. This means that M_0 is parametrized on Σ by the Weierstrass Representation formula:

$$\psi(z) = \operatorname{Re} \int_{z_0}^{z} \left(\tfrac{1}{2}(1-g^2)\omega, \tfrac{i}{2}(1+g^2)\omega, g\omega\right) \tag{1}$$

Without loss of generality, we can assume that $\Sigma = \mathbb{C} \cup \{\infty\} \setminus \{p_1, \cdots, p_n\}$, where $p_1, \cdots, p_n$ are complex numbers and $g \neq 0, \infty$ at $p_1, \cdots, p_n$ (by rotating M_0 if necessary). Then ω needs a double pole at $p_1, \cdots, p_n$ so has $2n-2$ zeros, counting multiplicity. Since ω needs a zero at each pole of g, with twice the multiplicity, it follows that g has $n-1$ poles so has degree $n-1$. Hence we may write

$$g = \frac{A(z)}{B(z)} \quad \text{and} \quad \omega = \frac{B(z)^2\, dz}{\prod_{i=1}^n (z-p_i)^2} \tag{2}$$

where

$$A(z) = \sum_{i=1}^n a_i z^{n-i} \quad \text{and} \quad B(z) = \sum_{i=1}^n b_i z^{n-i}.$$

We are going to deform this Weierstrass data, so we see a_i, b_i and p_i for $1 \leq i \leq n$ as complex parameters. We denote by $\mathbf{x} \in \mathbb{C}^{3n}$ the vector of these parameters, and by $\mathbf{x}_0$ the value of the parameters corresponding to the minimal n-noid M_0.

Let γ_i be the homology class of a small circle centered at p_i and define the following periods for $1 \leq i \leq n$ and $0 \leq k \leq 2$, depending on the parameter vector $\mathbf{x} \in \mathbb{C}^{3n}$:

$$P_{i,k}(\mathbf{x}) = \int_{\gamma_i} g^k \omega$$

$$P_i(\mathbf{x}) = (P_{i,0}(\mathbf{x}), P_{i,1}(\mathbf{x}), P_{i,2}(\mathbf{x})) \in \mathbb{C}^3$$

$$Q_i(\mathbf{x}) = \int_{\gamma_i} \left(\tfrac{1}{2}(1-g^2)\omega, \tfrac{\mathrm{i}}{2}(1+g^2)\omega, g\omega\right) \in \mathbb{C}^3.$$

Then

$$Q_i(\mathbf{x}) = \left(\tfrac{1}{2}(P_{i,0(\mathbf{x})} - P_{i,2}(\mathbf{x})), \tfrac{\mathrm{i}}{2}(P_{i,0}(\mathbf{x}) + P_{i,2}(\mathbf{x})), P_{i,1}(\mathbf{x})\right).$$

The components of $Q_i(\mathbf{x}_0)$ are imaginary because the Period Problem is solved for M_0. This gives

$$P_{i,2}(\mathbf{x}_0) = \overline{P_{i,0}(\mathbf{x}_0)} \quad \text{and} \quad P_{i,1}(\mathbf{x}_0) \in \mathrm{i}\mathbb{R} \tag{3}$$

Moreover, $\mathrm{Im}(Q_i(\mathbf{x}_0)) = -\phi_i$ where ϕ_i is the flux vector of M_0 at the end p_i. By the Residue Theorem, we have for all $\mathbf{x}$ in a neighborhood of $\mathbf{x}_0$:

$$\sum_{i=1}^n P_i(\mathbf{x}) = 0$$

Let $P = (P_1, \cdots, P_{n-1})$ and $Q = (Q_1, \cdots, Q_{n-1})$.

Definition 1. *M_0 is non-degenerate if the differential of P (or equivalently, Q) at $\mathbf{x}_0$ has complex rank $3n-3$.*

Remark 2. *If $n \geq 3$, we may (using Möbius transformations of the sphere) fix the value of three points, say p_1, p_2, p_3. Then "non-degenerate" means that the differential of P with respect to the remaining parameters is an isomorphism of $\mathbb{C}^{3n-3}$.*

This notion is related to another standard notion of non-degeneracy:

Definition 2. *M_0 is non-degenerate if its space of bounded Jacobi fields has (real) dimension 3.*

Theorem 2. *If M_0 is non-degenerate in the sense of Definition 2, then M_0 is non-degenerate in the sense of Definition 1.*

Proof. Assume M_0 is non-degenerate in the sense of Definition 2. Then in a neighborhood of M_0, the space $\mathcal{M}$ of minimal n-noids (up to translation) is a smooth manifold of dimension $3n-3$ by a standard application of the Implicit Function Theorem. Moreover, if we write $\phi_i \in \mathbb{R}^3$ for the flux vector at the i-th end, then the map $\phi = (\phi_1, \cdots, \phi_n)$ provides a local diffeomorphism between $\mathcal{M}$ and the space V of vectors $v = (v_1, \cdots, v_n) \in (\mathbb{R}^3)^n$ such that $\sum_{i=1}^n v_i = 0$. (All this is proved in Section 4 of [2] in the case where all ends are coplanar. The argument goes through in the general case.) Hence given a vector $v \in V$, there exists a deformation M_t of M_0 such that $M_t \in \mathcal{M}$ and $\frac{d}{dt}\phi(M_t)|_{t=0} = v$. We may write the Weierstrass data of M_t as above and obtain a set of parameters $\mathbf{x}(t)$, depending smoothly on t, such that $\mathbf{x}(0) = \mathbf{x}_0$. Then $dQ(\mathbf{x}_0) \cdot \mathbf{x}'(0) = -\mathrm{i}\, v$. Since Q is holomorphic, its differential is complex-linear so $dQ(\mathbf{x}_0)$ has complex rank equal to $\dim V = 3n - 3$. □

If all ends of M_0 have coplanar axes, then M_0 is non-degenerate in the sense of Definition 2 by Proposition 2 in [2]. In particular, the (most symmetric) n-noids of Jorge-Meeks are non-degenerate. This implies that generic n-noids in the component of the Jorge-Meeks n-noid are non-degenerate.

3. Background

In this section, we recall standard notations and results used in the DPW method. We work in the "untwisted" setting.

3.1. Loop groups.

A loop is a smooth map from the unit circle $\mathbb{S}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ to a matrix group. The circle variable is denoted λ and called the spectral parameter. The unit disk is denoted $\mathbb{D}$. For $\rho > 1$, we denote $\mathbb{D}_\rho$ the disk $|\lambda| < \rho$, $\mathbb{D}_\rho^* = \mathbb{D}_\rho \setminus \{0\}$ and $\mathbb{A}_\rho$ the annulus $\frac{1}{\rho} < |\lambda| < \rho$.

- If G is a matrix Lie group (or Lie algebra), ΛG denotes the group (or algebra) of smooth maps $\Phi : \mathbb{S}^1 \to G$.
- $\Lambda_+ SL(2,\mathbb{C}) \subset \Lambda SL(2,\mathbb{C})$ is the subgroup of maps B which extend holomorphically to $\mathbb{D}$ with $B(0)$ upper triangular.
- $\Lambda_+^{\mathbb{R}} SL(2,\mathbb{C}) \subset \Lambda_+ SL(2,\mathbb{C})$ is the subgroup of maps B such that $B(0)$ has positive entries on the diagonal.

Theorem 3 (Iwasawa decomposition)**.** *The multiplication $\Lambda SU(2) \times \Lambda_+^{\mathbb{R}} SL(2,\mathbb{C}) \to \Lambda SL(2,\mathbb{C})$ is a diffeomorphism. The unique splitting of an element $\Phi \in \Lambda SL(2,\mathbb{C})$ as $\Phi = FB$ with $F \in \Lambda SU(2)$ and $B \in \Lambda_+^{\mathbb{R}} SL(2,\mathbb{C})$ is called Iwasawa decomposition. F is called the unitary factor of Φ and denoted* $\mathrm{Uni}(\Phi)$. *B is called the positive factor and denoted* $\mathrm{Pos}(\Phi)$.

3.2. The matrix model of $\mathbb{R}^3$.

In the DPW method, one identifies $\mathbb{R}^3$ with the Lie algebra $\mathfrak{su}(2)$ by

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3 \longleftrightarrow X = -\mathrm{i} \begin{pmatrix} -x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_3 \end{pmatrix} \in \mathfrak{su}(2).$$

We have $\det(X) = \|x\|^2$. The group $SU(2)$ acts as linear isometries on $\mathfrak{su}(2)$ by conjugation: $H \cdot X = HXH^{-1}$.

3.3. **The DPW method.** The input data for the DPW method is a quadruple $(\Sigma, \xi, z_0, \phi_0)$ where:

- Σ is a Riemann surface.
- $\xi = \xi(z,\lambda)$ is a $\Lambda\mathfrak{sl}(2,\mathbb{C})$-valued holomorphic 1-form on Σ called the DPW potential. More precisely,

$$\xi = \begin{pmatrix} \alpha & \lambda^{-1}\beta \\ \gamma & -\alpha \end{pmatrix} \tag{4}$$

  where $\alpha(z,\lambda)$, $\beta(z,\lambda)$, $\gamma(z,\lambda)$ are holomorphic 1-forms on Σ with respect to the z variable, and are holomorphic with respect to λ in the disk $\mathbb{D}_\rho$ for some $\rho > 1$.
- $z_0 \in \Sigma$ is a base point.
- $\phi_0 \in \Lambda SL(2,\mathbb{C})$ is an initial condition.

Given this data, the DPW method is the following procedure.

- Let $\widetilde{\Sigma}$ be the universal cover of Σ and $\widetilde{z}_0 \in \widetilde{\Sigma}$ be an arbitrary element in the fiber of z_0. Solve the Cauchy Problem on $\widetilde{\Sigma}$:

$$\begin{cases} d\Phi(z,\lambda) = \Phi(z,\lambda)\xi(z,\lambda) \\ \Phi(\widetilde{z}_0,\lambda) = \phi_0(\lambda) \end{cases} \tag{5}$$

  to obtain a solution $\Phi : \widetilde{\Sigma} \to \Lambda SL(2,\mathbb{C})$.
- Compute the Iwasawa decomposition $(F(z,\cdot), B(z,\cdot))$ of $\Phi(z,\cdot)$.
- Define $f : \widetilde{\Sigma} \to \mathfrak{su}(2) \sim \mathbb{R}^3$ by the Sym-Bobenko formula:

$$f(z) = -2\mathrm{i}\frac{\partial F}{\partial \lambda}(z,1)F(z,1)^{-1} =: \mathrm{Sym}(F(z,\cdot)). \tag{6}$$

  Then f is a CMC-1 (branched) conformal immersion. f is regular at z (meaning unbranched) if and only if $\beta(z,0) \neq 0$. Its Gauss map is given by

$$N(z) = -\mathrm{i}F(z,1)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}F(z,1)^{-1} =: \mathrm{Nor}(F(z,\cdot)). \tag{7}$$

  The DPW method actually constructs a moving frame for f. The differential of f is given by

$$df(z) = 2\mathrm{i}\,B_{11}(z,0)^2F(z,1)\begin{pmatrix} 0 & \beta(z,0) \\ \overline{\beta}(z,0) & 0 \end{pmatrix}F(z,1)^{-1}. \tag{8}$$

Equation (8) can also be obtained by differentiation of the Sym-Bobenko formula (6).

Remark 3. *In* [18], *I have opposite signs in Equations* (6) *and* (7). *This is unfortunate because it makes the basis* (f_x, f_y, N) *negatively oriented. Equation* (6) *is the right formula, which one obtains by untwisting the standard Sym-Bobenko formula in the twisted case. See* [13] *or* [17].

3.4. **The Monodromy Problem.** Assume that Σ is not simply connected so its universal cover $\widetilde{\Sigma}$ is not trivial. Let $\mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$ be the group of fiber-preserving diffeomorphisms of $\widetilde{\Sigma}$. Let Φ be the solution of the Cauchy Problem (5). For $\gamma \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$, let

$$\mathcal{M}_\gamma(\Phi)(\lambda) = \Phi(\gamma(z),\lambda)\Phi(z,\lambda)^{-1}$$

be the monodromy of Φ with respect to γ (which is independent of $z \in \widetilde{\Sigma}$). The standard condition which ensures that the immersion f descends to a well defined immersion on Σ is the following system of equations, called the Monodromy Problem.

$$\forall \gamma \in \text{Deck}(\widetilde{\Sigma}/\Sigma) \quad \begin{cases} \mathcal{M}_\gamma(\Phi) \in \Lambda SU(2) & (i) \\ \mathcal{M}_\gamma(\Phi)(1) = \pm I_2 & (ii) \\ \frac{\partial \mathcal{M}_\gamma(\Phi)}{\partial \lambda}(1) = 0 & (iii) \end{cases} \tag{9}$$

One can identify $\text{Deck}(\widetilde{\Sigma}/\Sigma)$ with the fundamental group $\pi_1(\Sigma, z_0)$ (see for example Theorem 5.6 in [5]), so we will in general see γ as an element of $\pi_1(\Sigma, z_0)$. Under this identification, the monodromy of Φ with respect to $\gamma \in \pi_1(\Sigma, z_0)$ is given by

$$\mathcal{M}_\gamma(\Phi)(\lambda) = \Phi(\widetilde{\gamma}(1), \lambda)\Phi(\widetilde{\gamma}(0), \lambda)^{-1}$$

where $\widetilde{\gamma} : [0,1] \to \widetilde{\Sigma}$ is the lift of γ such that $\widetilde{\gamma}(0) = \widetilde{z}_0$.

3.5. Gauging.

Definition 3. *A gauge on Σ is a map $G : \Sigma \to \Lambda_+ SL(2, \mathbb{C})$ such that $G(z, \lambda)$ depends holomorphically on $z \in \Sigma$ and $\lambda \in \mathbb{D}_\rho$ for some $\rho > 1$.*

Let Φ be a solution of $d\Phi = \Phi\xi$ and G be a gauge. Let $\widehat{\Phi} = \Phi \times G$. Then $\widehat{\Phi}$ and Φ define the same immersion f. This is called "gauging". The gauged potential is

$$\widehat{\xi} = \widehat{\Phi}^{-1} d\widehat{\Phi} = G^{-1}\xi G + G^{-1} dG$$

and will be denoted $\xi \cdot G$, the dot denoting the action of the gauge group on the potential.

3.6. **Functional spaces.** We need to introduce a functional space for functions on the unit circle. We need that space to be a Banach algebra, and functions in that space should extend holomorphically to a neighborhood of the unit circle. The following choice is natural. We decompose a smooth function $f : \mathbb{S}^1 \to \mathbb{C}$ in Fourier series

$$f(\lambda) = \sum_{i \in \mathbb{Z}} f_i \lambda^i$$

Fix some $\rho > 1$ and define

$$\|f\| = \sum_{i \in \mathbb{Z}} |f_i| \rho^{|i|}$$

Let $\mathcal{W}_\rho$ be the space of functions f with finite norm. This is a Banach algebra, owing to the fact that the weight $\rho^{|i|}$ is submultiplicative (see Section 4 in [8]). Functions in $\mathcal{W}_\rho$ extend holomorphically to the annulus $\mathbb{A}_\rho$.

We define $\mathcal{W}_\rho^{\geq 0}$, $\mathcal{W}_\rho^{>0}$, $\mathcal{W}_\rho^{\leq 0}$ and $\mathcal{W}_\rho^{<0}$ as the subspaces of functions f such that $f_i = 0$ for $i < 0$, $i \leq 0$, $i > 0$ and $i \geq 0$, respectively. Functions in $\mathcal{W}_\rho^{\geq 0}$ extend holomorphically to the disk $\mathbb{D}_\rho$ and satisfy $|f(\lambda)| \leq \|f\|$ for all $\lambda \in \mathbb{D}_\rho$. We write $\mathcal{W}^0 \sim \mathbb{C}$ for the subspace of constant functions, so we have a direct sum $\mathcal{W}_\rho = \mathcal{W}_\rho^{<0} \oplus \mathcal{W}^0 \oplus \mathcal{W}_\rho^{>0}$. (The Banach algebra $\mathcal{W}_\rho$ is said to be decomposable, see [7] page 70.) A function f will be decomposed as $f = f^- + f^0 + f^+$ with $(f^-, f^0, f^+) \in \mathcal{W}_\rho^{<0} \times \mathcal{W}^0 \times \mathcal{W}_\rho^{>0}$.

We define the star operator by

$$f^*(\lambda)=\overline{f(1/\overline{\lambda})}=\sum_{i\in\mathbb{Z}}\overline{f_{-i}}\lambda^i$$

The involution $f\mapsto f^*$ exchanges $\mathcal{W}_\rho^{\geq 0}$ and $\mathcal{W}_\rho^{\leq 0}$. We have $\lambda^*=\lambda^{-1}$ and $c^*=\overline{c}$ if c is a constant. A function f is real on the unit circle if and only if $f=f^*$. We extend the star involution to loops by $M^*(\lambda)=\overline{M(1/\overline{\lambda})}^T$, so a loop F is unitary if and only if $F^*F=I_2$.

If L is a loop group, we denote $L_\rho\subset L$ the subgroup of loops whose entries are in $\mathcal{W}_\rho$. The loop groups $\Lambda SL(2,\mathbb{C})_\rho$, $\Lambda SU(2)_\rho$ and $\Lambda_+^{\mathbb{R}}SL(2,\mathbb{C})_\rho$ are Banach Lie groups, moreover:

Proposition 1. *Iwasawa decomposition restricts to an analytic diffeomorphism between the Banach Lie groups $\Lambda SL(2,\mathbb{C})_\rho$ and $\Lambda SU(2)_\rho\times\Lambda_+^{\mathbb{R}}SL(2,\mathbb{C})_\rho$.*

Proof: let $\Phi\in\Lambda SL(2,\mathbb{C})_\rho$ and let (F,B) be its Iwasawa decomposition. We want to prove that F and B have entries in $\mathcal{W}_\rho$. Since F is unitary, we have $\Phi^*\Phi=B^*B$. Now B^* has only non-positive powers of λ, so (B^*,B) is a Birkhoff-type decomposition of $M=\Phi^*\Phi$. Since $M\in\Lambda SL(2,\mathbb{C})_\rho$, it is known that both factors of its Birkhoff decomposition have entries in $\mathcal{W}_\rho$, owing to the fact that $\mathcal{W}_\rho$ is decomposable (see Theorem 1.4 in [7]). So $B\in\Lambda_+^{\mathbb{R}}SL(2,\mathbb{C})_\rho$ and $F\in\Lambda SU(2)_\rho$ follows.

Let $\Lambda\mathfrak{sl}(2,\mathbb{C})_\rho$, $\Lambda\mathfrak{su}(2)_\rho$ and $\Lambda_+^{\mathbb{R}}\mathfrak{sl}(2,\mathbb{C})_\rho$ be the Banach Lie algebras of respectively $\Lambda SL(2,\mathbb{C})_\rho$, $\Lambda SU(2)_\rho$ and $\Lambda_+^{\mathbb{R}}SL(2,\mathbb{C})$. The following decomposition is standard (the factors can be written explicitly in term of Fourier coefficients):

$$\Lambda\mathfrak{sl}(2,\mathbb{C})_\rho=\Lambda\mathfrak{su}(2)_\rho\oplus\Lambda_+^{\mathbb{R}}\mathfrak{sl}(2,\mathbb{C})_\rho.$$

By the inverse mapping theorem, the multiplication $\Lambda SU(2)_\rho\times\Lambda_+^{\mathbb{R}}SL(2,\mathbb{C})_\rho\to\Lambda SL(2,\mathbb{C})_\rho$ is an analytic local diffeomorphism in a neighborhood of (I_2,I_2), and in a neighborhood of any element (F,B) using left multiplication by F and right multiplication by B. Since we already know the multiplication is bijective, it is a a global diffeomorphism. □

4. A blow-up result

In this section, we consider a one-parameter family of DPW potential ξ_t with solution Φ_t and assume that $\Phi_0(z,\lambda)$ is independent of λ. Then its unitary part $F_0(z,\lambda)$ is independent of λ. The Sym Bobenko formula yields that $f_0\equiv 0$, so the family f_t collapses to the origin as $t=0$. The following theorem says that the blow-up $\frac{1}{t}f_t$ converges to a minimal surface whose Weierstrass data is explicitly computed.

Theorem 4. *Let Σ be a Riemann surface, $(\xi_t)_{t\in I}$ a family of DPW potentials on Σ and $(\Phi_t)_{t\in I}$ a family of solutions of $d\Phi_t=\Phi_t\xi_t$ on the universal cover $\widetilde{\Sigma}$ of Σ, where $I\subset\mathbb{R}$ is a neighborhood of 0. Fix a base point $z_0\in\widetilde{\Sigma}$. Assume that*

(1) $(t,z)\mapsto\xi_t(z,\cdot)$ and $t\mapsto\Phi_t(z_0,\cdot)$ are C^1 maps into $\Lambda\mathfrak{sl}(2,\mathbb{C})_\rho$ and $\Lambda SL(2,\mathbb{C})_\rho$, respectively.
(2) For all $t\in I$, Φ_t solves the Monodromy Problem (9).
(3) $\Phi_0(z,\lambda)$ is independent of λ:

$$\Phi_0(z,\lambda)=\begin{pmatrix}a(z)&b(z)\\c(z)&d(z)\end{pmatrix}$$

Let $f_t = \mathrm{Sym}(\mathrm{Uni}(\Phi_t)) : \Sigma \to \mathbb{R}^3$ be the CMC-1 immersion given by the DPW method. Then

$$\lim_{t\to 0} \frac{1}{t} f_t(z) = \psi(z)$$

where $\psi : \Sigma \to \mathbb{R}^3$ is a (possibly branched) minimal immersion with the following Weierstrass data:

$$g(z) = \frac{-a(z)}{c(z)} \qquad \text{and} \qquad \omega = 4c(z)^2 \frac{\partial \xi^{(-1)}_{t;12}}{\partial t}|_{t=0}.$$

The limit is for the uniform C^1 convergence on compact subsets of Σ.

Here $\xi^{(-1)}_{t;12}$ denotes the coefficient of λ^{-1} in the upper right entry of ξ_t. In case $\omega = 0$, the minimal immersion degenerates into a point and ψ is constant.

Proof: by standard ODE theory, $(t,z) \mapsto \Phi_t(z,\cdot)$ is a C^1 map into $\Lambda SL(2,\mathbb{C})_\rho$. Let (F_t, B_t) be the Iwasawa decomposition of Φ_t. By Proposition 1, $(t,z) \mapsto F_t(z,\cdot)$ and $(t,z) \mapsto B_t(z,\cdot)$ are real analytic maps into $\Lambda SU(2)_\rho$ and $\Lambda^{\mathbb{R}}_+ SL(2,\mathbb{C})_\rho$, respectively. At $t = 0$, Φ_0 is constant with respect to λ, so its Iwasawa decomposition is the standard QR decomposition:

$$F_0 = \frac{1}{\sqrt{|a|^2 + |c|^2}} \begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \qquad B_0 = \frac{1}{\sqrt{|a|^2 + |c|^2}} \begin{pmatrix} |a|^2 + |c|^2 & \overline{a}b + \overline{c}d \\ 0 & 1 \end{pmatrix}.$$

The Sym-Bobenko formula (6) yields $f_0 = 0$. Let $\mu_t = B^0_{t;11}$ and $\beta_t = \xi^{(-1)}_{t;12}$. By Equation (8), we have

$$df_t(z) = 2\mathrm{i}\, \mu_t(z)^2 F_t(z,1) \begin{pmatrix} 0 & \beta_t(z) \\ \overline{\beta_t(z)} & 0 \end{pmatrix} F_t(z,1)^{-1}.$$

Hence $(t,z) \mapsto df_t(z)$ is a C^1 map. At $t = 0$, ξ_0 is constant with respect to λ, so $\beta_0 = 0$. Define $\widetilde{f}_t(z) = \frac{1}{t} f_t(z)$ for $t \neq 0$. Then $d\widetilde{f}_t(z)$ extends at $t = 0$, as a continous function of (t,z), by

$$\begin{aligned} d\widetilde{f}_0 = \frac{d}{dt} df_t|_{t=0} &= 2\mathrm{i} \begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \begin{pmatrix} 0 & \beta' \\ \overline{\beta'} & 0 \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ -c & a \end{pmatrix} \\ &= 2\mathrm{i} \begin{pmatrix} -ac\beta' - \overline{ac\beta'} & a^2\beta' - \overline{c^2\beta'} \\ \overline{a^2\beta'} - c^2\beta' & ac\beta' + \overline{ac\beta'} \end{pmatrix} \end{aligned}$$

where $\beta' = \frac{d}{dt}\beta_t|_{t=0}$. In euclidean coordinates, this gives

$$d\widetilde{f}_0 = 4\ \mathrm{Re}\left[\tfrac{1}{2}(c^2 - a^2)\beta', \tfrac{\mathrm{i}}{2}(c^2 + a^2)\beta', -ac\beta'\right].$$

Writing $g = \frac{-a}{c}$ and $\omega = 4c^2\beta'$, we obtain

$$\widetilde{f}_0(z) = \widetilde{f}_0(z_0) + \mathrm{Re} \int_{z_0}^{z} \left[\tfrac{1}{2}(1 - g^2)\omega, \tfrac{\mathrm{i}}{2}(1 + g^2)\omega, g\omega\right]$$

and we see that $\widetilde{f}_0$ is a minimal surface with Weierstrass data (g,ω). The last statement of Theorem 4 comes from the fact that $d\widetilde{f}_t$ converges uniformly to $d\widetilde{f}_0$ on compact subsets of Σ. □

4.1. **Example.** As an example, we consider the family of Delaunay surfaces given by the following DPW potential in $\mathbb{C}^*$:

$$\xi_t(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1}r+s \\ \lambda r+s & 0 \end{pmatrix}\frac{dz}{z} \quad \text{with} \quad \begin{cases} r+s=\frac{1}{2} \\ rs=t \\ r<s \end{cases}$$

with initial condition $\Phi_t(1)=I_2$. As $t\to 0$, we have $(r,s)\to(0,\frac{1}{2})$. We have

$$\Phi_0(z,\lambda) = \exp\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\log z = \frac{1}{2\sqrt{z}}\begin{pmatrix} z+1 & z-1 \\ z-1 & z+1 \end{pmatrix}$$

$$\frac{\partial\xi_t}{\partial t}|_{t=0} = \begin{pmatrix} 0 & 2\lambda^{-1} \\ 2\lambda & 0 \end{pmatrix}\frac{dz}{z}.$$

Theorem 4 applies and gives

$$g(z) = \frac{1+z}{1-z} \quad \text{and} \quad \omega(z) = 4\left(\frac{z-1}{2\sqrt{z}}\right)^2\frac{2\,dz}{z} = 2\left(\frac{z-1}{z}\right)^2 dz.$$

This is the Weierstrass data of a horizontal catenoid of waist-radius 4 and axis Ox_1, with $x_1\to+\infty$ at the end $z=0$.

5. The DPW potential

We now start the proof of Theorem 1. Let (g,ω) be the Weierstrass data of the given minimal n-noid M_0, written as in Section 2. We introduce $3n$ λ-dependent parameters a_i, b_i and p_i for $1\leq i\leq n$ in the functional space $\mathcal{W}_\rho^{\geq 0}$. The vector of these parameters is denoted $\mathbf{x}\in(\mathcal{W}_\rho^{\geq 0})^{3n}$. The parameter $\mathbf{x}$ is in a neighborhood of a (constant) central value $\mathbf{x}_0\in(\mathcal{W}^0)^{3n}$ which correspond to the Weierstrass data of M_0, written as in Section 2. We define

$$A_{\mathbf{x}}(z,\lambda) = \sum_{i=1}^n a_i(\lambda)z^{n-i}$$

$$B_{\mathbf{x}}(z,\lambda) = \sum_{i=1}^n b_i(\lambda)z^{n-i}$$

$$g_{\mathbf{x}}(z,\lambda) = \frac{A_{\mathbf{x}}(z,\lambda)}{B_{\mathbf{x}}(z,\lambda)} \tag{10}$$

$$\omega_{\mathbf{x}}(z,\lambda) = \frac{B_{\mathbf{x}}(z,\lambda)^2\,dz}{\prod_{i=1}^n(z-p_i(\lambda))^2}. \tag{11}$$

For t in a neighborhood of 0 in $\mathbb{R}$, we consider the following DPW potential:

$$\xi_{t,\mathbf{x}}(z,\lambda) = \begin{pmatrix} 0 & \frac{1}{4}t(\lambda-1)^2\lambda^{-1}\omega_{\mathbf{x}}(z,\lambda) \\ dg_{\mathbf{x}}(z,\lambda) & 0 \end{pmatrix}.$$

We fix a base point z_0, away from the poles of g and ω, and we take the initial condition

$$\phi_0(\lambda) = \begin{pmatrix} g_{\mathbf{x}}(z_0,\lambda) & 1 \\ -1 & 0 \end{pmatrix}.$$

These choices are motivated by the following observations:

(1) At $t=0$, we have
$$\xi_{0,\mathbf{x}}(z,\lambda)=\begin{pmatrix}0 & 0\\ dg_{\mathbf{x}}(z,\lambda) & 0\end{pmatrix}.$$
The solution of the Cauchy Problem (5) is given by
$$\Phi_{0,\mathbf{x}}(z,\lambda)=\begin{pmatrix}g_{\mathbf{x}}(z,\lambda) & 1\\ -1 & 0\end{pmatrix}\tag{12}$$
which is well-defined, so the Monodromy Problem (9) is solved at $t=0$.

(2) The same conclusion holds if $\lambda=1$ instead of $t=0$. In particular, Items (ii) and (iii) of the Monodromy Problem (9) are automatically solved.

(3) At $\mathbf{x}=\mathbf{x}_0$, we have $g_{\mathbf{x}_0}=g$ so $\Phi_{0,\mathbf{x}_0}(z,\lambda)$ is independent of λ. Moreover,
$$\frac{\partial\xi^{(-1)}_{t,\mathbf{x}_0;12}}{\partial t}|_{t=0}=\frac{\omega}{4}.$$
Provided the Monodromy Problem is solved for all t in a neighborhood of 0, Theorem 4 applies and the limit minimal surface has Weierstrass data (g,ω) so is the minimal n-noid M_0, up to translation (see details in Section 7.1).

Remark 4. *The potential $\xi_{t,\mathbf{x}}$ is inspired from the potential used in* [18] *to construct CMC n-noids by perturbation of a sphere. In fact, in the case $dg_{\mathbf{x}}=dz$, the two potentials are dual to each other. (See Section 3.2.8 of* [19] *for the definition of duality in the DPW method.)*

5.1. **Regularity.** Our potential $\xi_{t,\mathbf{x}}$ has poles at the zeros of $B_{\mathbf{x}}$ and the points $p_1,\cdots,p_n$. (At ∞, we have $\omega_{\mathbf{x}}\sim b_1^2z^{-2}dz$ which is holomorphic.) We want the zeros of $B_{\mathbf{x}}$ to be apparent singularities, so we require the potential to be gauge-equivalent to a regular potential in a neighborhood of these points. Consider the gauge
$$G_{\mathbf{x}}(z,\lambda)=\begin{pmatrix}g_{\mathbf{x}}(z,\lambda)^{-1} & -1\\ 0 & g_{\mathbf{x}}(z,\lambda)\end{pmatrix}$$
The gauged potential is
$$\widehat{\xi}_{t,\mathbf{x}}:=\xi_{t,\mathbf{x}}\cdot G_{\mathbf{x}}=\begin{pmatrix}0 & \frac{1}{4}t(\lambda-1)^2\lambda^{-1}g_{\mathbf{x}}^2\omega_{\mathbf{x}}\\ g_{\mathbf{x}}^{-2}dg_{\mathbf{x}} & 0\end{pmatrix}.$$
We have
$$g_{\mathbf{x}}^{-2}dg_{\mathbf{x}}=\frac{A'_{\mathbf{x}}B_{\mathbf{x}}-A_{\mathbf{x}}B'_{\mathbf{x}}}{A_{\mathbf{x}}^2}\quad\text{and}\quad g_{\mathbf{x}}^2\omega_{\mathbf{x}}=\frac{A_{\mathbf{x}}^2\,dz}{\prod_{i=1}^n(z-p_i)^2}.$$
Let ζ be a zero of $B_{\mathbf{x}_0}$ (recall that $B_{\mathbf{x}_0}$ does not depend on λ). Then $A_{\mathbf{x}_0}(\zeta)\neq 0$. By continuity, there exists a neighborhood U of ζ such that for $z\in U$, $\lambda\in\mathbb{D}_\rho$ and $\mathbf{x}$ close enough to $\mathbf{x}_0$, $A_{\mathbf{x}}(z,\lambda)\neq 0$. So $\widehat{\xi}_{t,\mathbf{x}}$ is holomorphic in $U\times\mathbb{D}_\rho^*$ and moreover, $\widehat{\xi}^{(-1)}_{t,\mathbf{x};12}\neq 0$. This ensures that the immersion extends analytically to U and is unbranched in U.

6. The monodromy problem

6.1. **Formulation of the problem.** For $i\in[1,n]$, we denote $p_{i,0}$ the central value of the parameter p_i (so $p_{1,0},\cdots,p_{n,0}$ are the ends of the minimal n-noid M_0). We consider the following λ-independent domain on the Riemann sphere:
$$\Omega=\{z\in\mathbb{C}:\forall i\in[1,n],|z-p_{i,0}|>\varepsilon\}\cup\{\infty\}\tag{13}$$

where $\varepsilon > 0$ is a fixed, small enough number such that the disks $D(p_{i,0}, 8\varepsilon)$ for $1 \le i \le n$ are disjoint. As in [18], we first construct a family of immersions f_t on Ω. Then we extend f_t to an n-punctured sphere in Proposition 4.

Let $\widetilde{\Omega}$ be the universal cover of Ω and $\Phi_{t,\mathbf{x}}(z,\lambda)$ be the solution of the following Cauchy Problem on $\widetilde{\Omega}$:

$$\left\{ \begin{array}{l} d\Phi_{t,\mathbf{x}}(z,\lambda) = \Phi_{t,\mathbf{x}}(z,\lambda)\xi_{t,\mathbf{x}}(z,\lambda) \\ \Phi_{t,\mathbf{x}}(\widetilde{z}_0,\lambda) = \phi_0 \end{array} \right. \tag{14}$$

We denote $\gamma_1, \cdots, \gamma_{n-1}$ a set of generators of the fundamental group $\pi_1(\Omega, z_0)$, with γ_i encircling the point $p_{i,0}$. We may assume that each γ_i is represented by a fixed curve avoiding the poles of $\xi_{t,\mathbf{x}}$. Let

$$M_i(t,\mathbf{x}) = \mathcal{M}_{\gamma_i}(\Phi_{t,\mathbf{x}})$$

be the monodromy of $\Phi_{t,\mathbf{x}}$ along γ_i. By Equation (12), we have $M_i(0,\mathbf{x}) = I_2$. Recall that the matrix exponential is a local diffeomorphism from a neighborhood of 0 in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ (respectively $\mathfrak{su}(2)$) to a neighborhood of I_2 in $SL(2,\mathbb{C})$ (respectively $SU(2)$). The inverse diffeomorphism is denoted log. For $t \neq 0$ small enough and $\lambda \in \mathbb{D}_\rho \setminus \{1\}$, we define as in [18]

$$\widetilde{M}_i(t,\mathbf{x})(\lambda) = \frac{4\lambda}{t(\lambda-1)^2} \log M_i(t,\mathbf{x})(\lambda).$$

Proposition 2. *(1) $\widetilde{M}_i(t,\mathbf{x})(\lambda)$ extends smoothly at $t = 0$ and $\lambda = 1$, and each entry $\widetilde{M}_{i;k\ell}$ is a smooth map from a neighborhood of $(0,\mathbf{x}_0)$ in $\mathbb{R} \times (\mathcal{W}_\rho^{\ge 0})^3$ to $\mathcal{W}_\rho$.*

(2) At $t = 0$, we have

$$\widetilde{M}_i(0,\mathbf{x})(\lambda) = \begin{pmatrix} \mathcal{P}_{i,1}(\mathbf{x}) & \mathcal{P}_{i,2}(\mathbf{x}) \\ -\mathcal{P}_{i,0}(\mathbf{x}) & -\mathcal{P}_{i,1}(\mathbf{x}) \end{pmatrix} \quad \textit{where } \mathcal{P}_{i,k}(\mathbf{x}) = \int_{\gamma_i} g_{\mathbf{x}}^k \omega_{\mathbf{x}}. \tag{15}$$

(3) The Monodromy Problem (9) is equivalent to

$$\widetilde{M}_i(t,\mathbf{x}) \in \Lambda\mathfrak{su}(2) \quad \textit{for} \quad 1 \le i \le n-1. \tag{16}$$

Proof: we follow the proof of Proposition 1 in [18]. We first consider the case where the parameter $\mathbf{x} = (a_i, b_i, p_i)_{1 \le i \le n}$ is constant with respect to λ, so $\mathbf{x} \in \mathbb{C}^{3n}$. For $(\mu, \mathbf{x})$ in a neighborhood of $(0, \mathbf{x}_0)$ in $\mathbb{C} \times \mathbb{C}^{3n}$, we define

$$\widehat{\xi}_{\mu,\mathbf{x}}(z) = \begin{pmatrix} 0 & \mu\,\omega_{\mathbf{x}}(z) \\ dg_{\mathbf{x}}(z) & 0 \end{pmatrix}$$

where $\omega_{\mathbf{x}}$ and $g_{\mathbf{x}}$ are defined by Equations (10) and (11), except that a_i, b_i, p_i are constant complex numbers. Let $\widehat{\Phi}_{\mu,\mathbf{x}}$ be the solution of the Cauchy Problem $d\widehat{\Phi}_{\mu,\mathbf{x}} = \widehat{\Phi}_{\mu,\mathbf{x}}\widehat{\xi}_{\mu,\mathbf{x}}$ in $\widetilde{\Omega}$ with initial condition $\widehat{\Phi}_{\mu,\mathbf{x}}(\widetilde{z}_0) = \phi_0$. Let $N_i(\mu,\mathbf{x}) = \mathcal{M}_{\gamma_i}(\widehat{\Phi}_{\mu,\mathbf{x}})$. By standard ODE theory, each entry of N_i is a holomorphic function of $(\mu,\mathbf{x})$. At $\mu = 0$, $\widehat{\Phi}_{0,\mathbf{x}}$ is given by Equation (12), so in particular $N_i(0,\mathbf{x}) = I_2$. Hence

$$\widetilde{N}_i(\mu,\mathbf{x}) := \frac{1}{\mu} \log N_i(\mu,\mathbf{x})$$

extends holomorphically at $\mu = 0$ with value $\widetilde{N}_i(0,\mathbf{x}) = \frac{\partial N_i}{\partial \mu}(0,\mathbf{x})$. By Proposition 8 in Appendix A of [18] (the same formula appeared before on page 39 of [12]):

$$\frac{\partial N_i}{\partial \mu}(0,\mathbf{x}) = \int_{\gamma_i} \widehat{\Phi}_{0,\mathbf{x}} \frac{\partial \widehat{\xi}_{\mu,\mathbf{x}}}{\partial \mu}|_{\mu=0} \widehat{\Phi}_{0,\mathbf{x}}^{-1}.$$

Hence

$$\widetilde{N}_i(0,\mathbf{x}) = \int_{\gamma_i} \begin{pmatrix} g_{\mathbf{x}} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_{\mathbf{x}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & g_{\mathbf{x}} \end{pmatrix} = \int_{\gamma_i} \begin{pmatrix} g_{\mathbf{x}}\omega_{\mathbf{x}} & g_{\mathbf{x}}^2\omega_{\mathbf{x}} \\ -\omega_{\mathbf{x}} & -g_{\mathbf{x}}\omega_{\mathbf{x}} \end{pmatrix}. \tag{17}$$

For $(t,\mathbf{x})$ in a neighborhood of $(0,\mathbf{x}_0)$ in $\mathbb{R} \times (\mathcal{W}_\rho^{\geq 0})^{3n}$, we have

$$\xi_{t,\mathbf{x}}(z,\lambda) = \widehat{\xi}_{\mu(t,\lambda),\mathbf{x}(\lambda)}(z) \quad \text{with} \quad \mu(t,\lambda) = \frac{t(\lambda-1)^2}{4\lambda}.$$

Hence

$$M_i(t,\mathbf{x})(\lambda) = N_i(\mu(t,\lambda),\mathbf{x}(\lambda)) \quad \text{and} \quad \widetilde{M}_i(t,\mathbf{x})(\lambda) = \widetilde{N}_i(\mu(t,\lambda),\mathbf{x}(\lambda)).$$

By substitution (see Proposition 9 in Appendix B of [18]), each entry of $\widetilde{M}_i$ is is a smooth map from a neighborhood of $(0,\mathbf{x}_0)$ in $\mathbb{R} \times (\mathcal{W}_\rho^{\geq 0})^3$ to $\mathcal{W}_\rho$. Moreover, $\widetilde{M}_i(0,\mathbf{x})$ is given by Equation (17). The fact that $\widetilde{M}_i$ extends holomorphically at $\lambda = 1$ implies that Points (ii) and (iii) of Problem (9) are automatically satisfied. Since $\lambda^{-1}(\lambda-1)^2 \in \mathbb{R}$ for $\lambda \in \mathbb{S}^1$, Equation (i) of Problem (9) is equivalent to Equation (16). □

6.2. **Solution of the monodromy problem.** Without loss of generality, we may (using a Möbius transformation of the sphere) fix the value of p_1, p_2 and p_3. We still denote $\mathbf{x} \in (\mathcal{W}_\rho^{\geq 0})^{3n-3}$ the vector of the remaining parameters.

Proposition 3. *Assume that the given minimal n-noid is non-degenerate. For t in a neighborhood of* 0*, there exists a smooth function* $\mathbf{x}(t) \in (\mathcal{W}_\rho^{\geq 0})^{3n-3}$ *such that* $\widetilde{M}_i(t,\mathbf{x}(t),\cdot) \in \Lambda\mathfrak{su}(2)$ *for* $1 \leq i \leq n-1$*. Moreover,* $\mathbf{x}(0) = \mathbf{x}_0$*.*

Proof: recalling the definition of $P_{i,k}$ in Section 2 and $\mathcal{P}_{i,k}$ in Equation (15), we have

$$\mathcal{P}_{i,k}(\mathbf{x})(\lambda) = P_{i,k}(\mathbf{x}(\lambda)).$$

Hence $\mathcal{P}_{i,k}$ is a smooth map from a neighborhood of $\mathbf{x}_0$ in $(\mathcal{W}_\rho^{\geq 0})^{3n-3}$ to $\mathcal{W}_\rho^{\geq 0}$. Moreover, since $\mathbf{x}_0$ is constant, we have for $X \in (\mathcal{W}_\rho^{\geq 0})^{3n-3}$:

$$(d\mathcal{P}_{i,k}(\mathbf{x}_0)X)(\lambda) = dP_{i,k}(\mathbf{x}_0)X(\lambda). \tag{18}$$

Let $\mathcal{P}_i = (\mathcal{P}_{i,0}, \mathcal{P}_{i,1}, \mathcal{P}_{i,2})$ and $\mathcal{P} = (\mathcal{P}_1, \cdots, \mathcal{P}_{n-1})$. By the non-degeneracy hypothesis and Remark 2, $dP(\mathbf{x}_0)$ is an automorphism of $\mathbb{C}^{3n-3}$, so $d\mathcal{P}(\mathbf{x}_0)$ is an automorphism of $(\mathcal{W}_\rho^{\geq 0})^{3n-3}$ and restricts to an automorphism of $(\mathcal{W}_\rho^{>0})^{3n-3}$.

We define the following smooth maps with value in $\mathcal{W}_\rho$ (the star operator is defined in Section 3.6)

$$\mathcal{F}_i(t,\mathbf{x}) = \widetilde{M}_{i,11}(t,\mathbf{x}) + \widetilde{M}_{i,11}(t,\mathbf{x})^*$$

$$\mathcal{G}_i(t,\mathbf{x}) = \widetilde{M}_{i,12}(t,\mathbf{x}) + \widetilde{M}_{i,21}(t,\mathbf{x})^*$$

Problem (16) is equivalent to $\mathcal{F}_i = \mathcal{G}_i = 0$. Actually, by definition, $\mathcal{F}_i = \mathcal{F}_i^*$, so Problem (16) is equivalent to

$$\mathcal{F}_i(t,\mathbf{x})^+ = 0, \quad \mathrm{Re}(\mathcal{F}_i(t,\mathbf{x})^0) = 0 \quad \text{ and } \mathcal{G}_i(t,\mathbf{x}) = 0 \quad \text{ for } 1 \le i \le n-1.$$

At $t = 0$, we have by Equation (15):

$$\mathcal{F}_i(0,\mathbf{x}) = \mathcal{P}_{i,1}(\mathbf{x}) + \mathcal{P}_{i,1}(\mathbf{x})^*$$

$$\mathcal{G}_i(0,\mathbf{x}) = \mathcal{P}_{i,2}(\mathbf{x}) - \mathcal{P}_{i,0}(\mathbf{x})^*$$

Equation (3) tells us precisely that that at the central value, we have $\mathcal{F}_i(0,\mathbf{x}_0) = 0$ and $\mathcal{G}_i(0,\mathbf{x}_0) = 0$. We have for $X \in (\mathcal{W}_\rho^{\ge 0})^{3n-3}$:

$$d\mathcal{F}_i(0,\mathbf{x}_0)X = d\mathcal{P}_{i,1}(x_0)X + (d\mathcal{P}_{i,1}(\mathbf{x}_0)X)^*$$

$$d\mathcal{G}_i(0,\mathbf{x}_0)X = d\mathcal{P}_{i,2}(x_0)X - (d\mathcal{P}_{i,0}(\mathbf{x}_0)X)^*$$

Projecting on $\mathcal{W}_\rho^{>0}$ and $\mathcal{W}_\rho^{<0}$ we obtain:

$$(d\mathcal{F}_i(0,\mathbf{x}_0)X)^+ = d\mathcal{P}_{i,1}(\mathbf{x}_0)X^+$$

$$(d\mathcal{G}_i(0,\mathbf{x}_0)X)^+ = d\mathcal{P}_{i,2}(\mathbf{x}_0)X^+$$

$$(d\mathcal{G}_i(0,\mathbf{x}_0)X)^- = -(d\mathcal{P}_{i,0}(\mathbf{x}_0)X^+)^*$$

$$(d\mathcal{G}_i(0,\mathbf{x}_0)X)^{-*} = -d\mathcal{P}_{i,0}(\mathbf{x}_0)X^+.$$

Hence the operator

$$\left[d\mathcal{F}_i(0,\mathbf{x}_0)^+, d\mathcal{G}_i(0,\mathbf{x}_0)^+, d\mathcal{G}_i(0,\mathbf{x}_0)^{-*}\right]_{1\le i\le n-1}$$

only depends on X^+ and is an automorphism of $(\mathcal{W}_\rho^{>0})^{3n-3}$ because $d\mathcal{P}(\mathbf{x}_0)$ is. Projecting on $\mathcal{W}^0$ we obtain:

$$(d\mathcal{F}_i(0,\mathbf{x}_0)X)^0 = 2\,\mathrm{Re}\left(d\mathcal{P}_{i,1}(\mathbf{x}_0)X^0\right)$$

$$(d\mathcal{G}_i(0,\mathbf{x}_0)X)^0 = d\mathcal{P}_{i,2}(\mathbf{x}_0)X^0 - \overline{d\mathcal{P}_{i,0}(\mathbf{x}_0)X^0}.$$

Hence the $\mathbb{R}$-linear operator

$$\left[\mathrm{Re}(d\mathcal{F}_i(0,\mathbf{x}_0)^0), d\mathcal{G}_i(0,\mathbf{x}_0)^0\right]_{1\le i\le n-1}$$

only depends on X^0 and is surjective from $\mathbb{C}^{3n-3}$ to $(\mathbb{R}\times\mathbb{C})^{3n-3}$. This implies that the differential of the map $(\mathcal{F}_i^+, \mathcal{G}_i^+, \mathcal{G}_i^{-*}, \mathrm{Re}(\mathcal{F}_i^0), \mathcal{G}_i^0)_{1\le i\le n-1}$ is surjective from $(\mathcal{W}_\rho^{\ge 0})^{3n-3}$ to $((\mathcal{W}_\rho^{>0})^3 \times \mathbb{R} \times \mathbb{C})^{n-1}$. Proposition 3 follows from the Implicit Function Theorem. □

Remark 5. *The kernel of the differential has real dimension $3n-3$ so we have $3n-3$ free real parameters. These parameters correspond to deformations of the flux vectors of the minimal n-noid.*

7. Geometry of the immersion

From now on, we assume that $\mathbf{x}(t)$ is given by Proposition 3. We write $a_{i,t}$, $b_{i,t}$ and $p_{i,t}$ for the value of the corresponding parameters. (These parameters are in the space $\mathcal{W}_\rho^{\geq 0}$ so are functions of λ.) For ease of notation, we write g_t, ω_t, ξ_t and Φ_t for $g_{\mathbf{x}(t)}$, $\omega_{\mathbf{x}(t)}$, $\xi_{t,\mathbf{x}(t)}$ and $\Phi_{t,\mathbf{x}(t)}$, respectively. Let $F_t = \text{Uni}(\Phi_t)$. Since the Monodromy Problem is solved, the Sym-Bobenko formula (6) defines a CMC-1 immersion $f_t : \Omega \to \mathbb{R}^3$, where Ω is the (fixed) domain defined by Equation (13).

Proposition 4. *The immersion f_t extends analytically to*

$$\Sigma_t := \mathbb{C} \cup \{\infty\} \setminus \{p_{1,t}(0), \cdots, p_{n,t}(0)\}$$

where $p_{i,t}(0)$ is the value of $p_{i,t}$ at $\lambda = 0$.

We omit the proof which is exactly the same as the proof of Point 1 of Proposition 4 in [18]. It relies on Theorem 3 in [18] which allows for λ-dependent changes of variables in the DPW method.

7.1. Convergence to the minimal n-noid.

Proposition 5. $\lim_{t\to 0} \frac{1}{t} f_t = \psi$ *where ψ is (up to translation) the conformal parametrization of the minimal n-noid given by Equation* (1). *The limit is the uniform C^1 convergence on compact subsets of $\Sigma_0 = \mathbb{C} \cup \{\infty\} \setminus \{p_{1,0}, \cdots, p_{n,0}\}$.*

Proof: at $t = 0$, we have $g_0 = g$ and $\omega_0 = \omega$. By Equation (12) and definition of the potential, we have

$$\Phi_0(z,\lambda) = \begin{pmatrix} g(z) & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial \xi_{t;12}^{(-1)}}{\partial t}|_{t=0} = \frac{\omega}{4}.$$

By Theorem 4, $\frac{1}{t} f_t$ converges to a minimal surface with Weierstrass data (g, ω) on compact subsets of Σ_0 minus the poles of g. In a neighborhood of the poles of g, we use the gauge introduced in Section 5.1. With the notations of this section and writing $\widehat{\Phi}_t = \Phi_t G_{\mathbf{x}(t)}$, we have

$$\widehat{\Phi}_0(z,\lambda) = \begin{pmatrix} 1 & 0 \\ -g(z)^{-1} & 1 \end{pmatrix} \quad \text{and} \quad \frac{\partial \widehat{\xi}_{t;12}^{(-1)}}{\partial t}|_{t=0} = \frac{g^2 \omega}{4}.$$

By Theorem 4 again, $\frac{1}{t} f_t$ converges to a minimal surface with Weierstrass data (g, ω) in a neighborhood of the poles of g. The two limit minimal surfaces are of course the same, since they coincide in a neighborhood of z_0. □

7.2. Delaunay ends.

In this section, we prove that the immersion f_t has Delaunay ends. Delaunay ends in the DPW method have been studied in [4, 13]. Following [17], we gauge our potential to a perturbation of the standard Delaunay potential and we use the results in [13].

We denote N_0 the Gauss map of the minimal n-noid M_0. For $1 \leq i \leq n$, we denote $\mathcal{C}_i$ the catenoid to which M_0 is asymptotic at $p_{i,0}$ and $\tau_i > 0$ the necksize of $\mathcal{C}_i$.

Definition 4. *We say that N_0 points to the inside in a neighborhood of $p_{i,0}$ if it points to the component of $\mathbb{R}^3 \setminus \mathcal{C}_i$ containing the axis of $\mathcal{C}_i$.*

Proposition 6. *For $1 \leq i \leq n$ and $t \neq 0$:*

(1) *The immersion f_t has a Delaunay end at $p_{i,t}$. If we denote $w_{i,t}$ its weight then*

$$\lim_{t\to 0} t^{-1} w_{i,t} = \pm 2\pi\tau_i$$

where the sign is + if N_0 points to the inside in a neighborhood of $p_{i,0}$ and − otherwise.

(2) *Its axis converges as $t \to 0$ to the half-line through the origin directed by the vector $N_0(p_{i,0})$.*

(3) *If N_0 points to the inside in a neighborhood of $p_{i,0}$, there exists a uniform $\varepsilon > 0$ such that for $t > 0$ small enough, $f_t(D^*(p_{i,0}, \varepsilon))$ is embedded.*

Proof: in a neighborhood of the puncture $p_{i,t}$, we may use $w = g_t(z) - g_t(p_{i,t})$ as a local coordinate. Note that $p_{i,t} \in \mathcal{W}_\rho^{\geq 0}$ so, as a function of λ, extends holomorphically to $\mathbb{D}_\rho$. Thus the coordinate w depends holomorphically on $\lambda \in \mathbb{D}_\rho$. This is not a problem by Theorem 3 in [18]. Consider the gauge

$$G(w) = \begin{pmatrix} \frac{k}{\sqrt{w}} & \frac{-1}{2k\sqrt{w}} \\ 0 & \frac{\sqrt{w}}{k} \end{pmatrix}.$$

Here we can take $k = 1$, but later on we will take another value of k so we do the computation for general values of $k \neq 0$. The gauged potential is

$$\widehat{\xi}_t := \xi_t \cdot G = \begin{pmatrix} 0 & \frac{dw}{4k^2 w} + \frac{wt(\lambda-1)^2}{4k^2\lambda}\omega_t \\ \frac{k^2 dw}{w} & 0 \end{pmatrix}$$

Since ω_t has a double pole at $p_{i,t}$, $\widehat{\xi}_t$ has a simple pole at $w = 0$ with residue

$$A_{i,t}(\lambda) = \begin{pmatrix} 0 & \frac{1}{4k^2} + \frac{t(\lambda-1)^2}{4k^2\lambda}\alpha_{i,t}(\lambda) \\ k^2 & 0 \end{pmatrix}$$

where

$$\alpha_{i,t} = \mathrm{Res}_{p_{i,t}}(w\omega_t) = \mathrm{Res}_{p_{i,t}}(g_t(z) - g_t(p_{i,t}))\omega_t. \tag{19}$$

Claim 1. *For t small enough, $\alpha_{i,t}$ is a real constant (i.e. independent of λ, possibly depending on t).*

Proof: the proof is similar to the proof of Point 2 of Proposition 4 in [18]. We use the standard theory of Fuchsian systems. Fix $t \neq 0$ and $\lambda \in \mathbb{S}^1 \setminus \{1\}$. Assume that $\alpha_{i,t}(\lambda) \neq 0$. Let $\widehat{\Phi}_t = \Phi_t G$. The eigenvalues of $A_{i,t}$ are $\pm\Lambda_{i,t}$ with

$$\Lambda_{i,t}(\lambda)^2 = \frac{1}{4} + \frac{t(\lambda-1)^2}{4\lambda}\alpha_{i,t}(\lambda).$$

Provided $t \neq 0$ is small enough, $\Lambda_{i,t} \notin \mathbb{Z}/2$ so the system is non resonant and $\widehat{\Phi}_t$ has the following standard $z^A P$ form in the universal cover of $D(0,\varepsilon)^*$:

$$\widehat{\Phi}_t(w,\lambda) = V(\lambda)\exp(A_{i,t}(\lambda)\log w)P(w,\lambda)$$

where $P(w,\lambda)$ descends to a well defined holomorphic function of $w \in D(0,\varepsilon)$ with $P(0,\lambda) = I_2$. Consequently, its monodromy is

$$\mathcal{M}_{\gamma_i}(\widehat{\Phi}_t) = V(\lambda)\exp(2\pi \mathrm{i} A_{i,t})V(\lambda)^{-1}$$

with eigenvalues $\exp(\pm 2\pi \mathrm{i}\Lambda_{i,t}(\lambda))$. Since the Monodromy Problem is solved, the eigenvalues are unitary complex numbers, so $\Lambda_{i,t}(\lambda)\in\mathbb{R}$ which implies that $\alpha_{i,t}(\lambda)\in\mathbb{R}$. This of course remains true if $\alpha_{i,t}(\lambda)=0$. Hence $\alpha_{i,t}$ is real on $\mathbb{S}^1\setminus\{1\}$. Since all the parameters involved in the definition of ω_t are in $\mathcal{W}_\rho^{\geq 0}$, $\alpha_{i,t}$ is holomorphic in the unit disk. Hence it is constant. □

Returning to the proof of Proposition 6, let $(r,s)\in\mathbb{R}^2$ be the solution of

$$\begin{cases} rs = \frac{1}{4}t\alpha_{i,t} \\ r+s=\frac{1}{2} \\ r<s \end{cases} \tag{20}$$

Since $r<s$, $\sqrt{r\lambda+s}$ is well defined and does not vanish for $\lambda\in\mathbb{D}$. We take $k=\sqrt{r\lambda+s}$ in the definition of the gauge G. Using Equation (20), we have:

$$(r\lambda^{-1}+s)(r\lambda+s)=\tfrac{1}{4}+rs(\lambda-1)^2\lambda^{-1}=\tfrac{1}{4}+\tfrac{1}{4}t(\lambda-1)^2\lambda^{-1}\alpha_{i,t}.$$

So the residue of $\widehat{\xi}_t$ becomes

$$A_{i,t}=\begin{pmatrix} 0 & \frac{1}{r\lambda+s}\left(\frac{1}{4}+\frac{t(\lambda-1)^2}{4\lambda}\alpha_{i,t}\right) \\ r\lambda+s & 0\end{pmatrix}=\begin{pmatrix} 0 & r\lambda^{-1}+s \\ r\lambda+s & 0\end{pmatrix}$$

which is the residue of the standard Delaunay potential. By [13], the immersion f_t has a Delaunay end at $p_{i,t}$ of weight $w_{i,t}=8\pi rs=2\pi t\alpha_{i,t}$. It remains to relate $\alpha_{i,0}$ to the logarithmic growth τ_i. For ease of notation, let us write $p_i=p_{i,0}$. Assume that N_0 points to the inside in a neighborhood of p_i. The flux of M_0 along γ_i is equal to

$$\phi_i=2\pi\tau_i\,N_0(p_i)=2\pi\frac{\tau_i}{|g(p_i)|^2+1}\left(2\operatorname{Re}(g(p_i)),2\operatorname{Im}(g(p_i)),|g(p_i)|^2-1\right)$$

On the other hand, we have seen in Section 2 that the flux is equal to

$$\phi_i=-2\pi\operatorname{Res}_{p_i}\left(\tfrac{1}{2}(1-g^2)\omega,\tfrac{\mathrm{i}}{2}(1+g^2)\omega,g\omega\right)$$

Comparing these two expressions for ϕ_i, we obtain

$$\operatorname{Res}_{p_i}(g\omega)=-\tau_i\frac{|g(p_i)|^2-1}{|g(p_i)|^2+1}\quad\text{and}\quad\operatorname{Res}_{p_i}\omega=-2\tau_i\frac{\overline{g(p_i)}}{|g(p_i)|^2+1}$$

Using Equation (19), this gives

$$\alpha_{i,0}=\operatorname{Res}_{p_i}(g\omega)-g(p_i)\operatorname{Res}_{p_i}\omega=\tau_i$$

If N_0 points to the outside in a neighborhood of p_i, then $\phi_i=-2\pi\tau_i\,N_0(p_i)$, so the same computation gives $\alpha_{i,0}=-\tau_i$. This proves Point 1 of Proposition 6.

To prove Point 2, we use Theorem 5 in Appendix A. We need to compute $\widehat{\Phi}_0$ at $w=1$. At $t=0$, we have $k=\frac{1}{\sqrt{2}}$ so

$$G(1)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}.$$

At $t=0$, we have $w=g(z)-g(p_i)$, so $w=1\Leftrightarrow g(z)=g(p_i)+1$. Using Equation (12),

$$\widehat{\Phi}_0(1)=\frac{1}{\sqrt{2}}\begin{pmatrix} g(p_i)+1 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} g(p_i)+1 & -g(p_i)+1\\ -1 & 1\end{pmatrix}.$$

Fix $0<\alpha<1$. By Theorem 5 (using $t\alpha_{i,t}$ as the time parameter), there exists $\varepsilon>0$, $T>0$ and c such that for $0<|t|<T$:

$$\|f_t(z)-f^{\mathcal{D}}_{i,t}(z)\|\leq c|t|\,|z-p_{i,t}|^{\alpha}\quad\text{ in } D^*(p_{i,t},\varepsilon)$$

where $f^{\mathcal{D}}_{i,t}:\mathbb{C}\setminus\{p_{i,t}\}\to\mathbb{R}^3$ is a Delaunay immersion. We compute the limit axis of $f^{\mathcal{D}}_{i,t}$ using Point 3 of Theorem 5:

$$\widehat{\Phi}_0(1)H=\begin{pmatrix} g(p_i) & 1\\ -1 & 0\end{pmatrix}=\Phi_0(p_i).$$
$$Q=F_0(p_i)$$
$$Qe_3Q^{-1}=\mathrm{Nor}(F_0(p_i))=N_0(p_i).$$

This proves Point 2 of Proposition 6. If N_0 points to the inside in a neighborhood of $p_{i,0}$, then for $t>0$, $t\alpha_{i,t}>0$ so Point 3 follows from Point 2 of Theorem 5. □

7.3. **Alexandrov-embeddedness.** We recall from [2, 9] the definition of Alexandrov-embeddedness in the non-compact case:

Definition 5. *A surface M of finite topology is Alexandrov-embedded if M is properly immersed, if each end of M is embedded, and if there exists a compact 3-manifold $\overline{W}$ with boundary $\partial\overline{W}=\overline{S}$, n points $q_1,\cdots,q_n\in\overline{S}$ and a proper immersion $F:W=\overline{W}\setminus\{q_1,\cdots,q_n\}\to\mathbb{R}^3$ whose restriction to $S=\overline{S}\setminus\{q_1,\cdots,q_n\}$ parametrizes M.*

Lemma 1. *Let M be an Alexandrov-embedded minimal surface with n catenoidal ends. With the notations of Definition 5, we equip W with the flat metric induced by F, so F is a local isometry, and we denote N the inside normal to S. Then there exists a flat 3-manifold W' containing W, a local isometry $F':W'\to\mathbb{R}^3$ extending F and $r>0$ such that the tubular neighborhood Tub_rS is embedded in W'. In other words, the map $(x,s)\mapsto\exp_x(sN(x))$ from $S\times(-r,r)$ to W' is well defined and is a diffeomorphism onto its image.*

Proof: since M has catenoidal ends, there exists $r>0$ such that the inside tubular neighborhood map

$$g:S\times(0,r)\to W,\quad g(x,s)=\exp_x(sN(x))$$

is a diffeomorphism onto its image. Since F is a local isometry, we have

$$F(g(x,s))=F(x)+s\,dF(x)N(x)\quad\text{ for } (x,s)\in S\times(0,r).\tag{21}$$

We define W' as the disjoint union $(S\times(-r,r))\sqcup W$ where we identify $(x,s)\in S\times(0,r)$ with its image $g(x,s)\in W$. We define $F':W'\to\mathbb{R}^3$ by $F'=F$ in W and

$$F'(x,s)=F(x)+s\,dF(x)N(x)\quad\text{ for } (x,s)\in S\times(-r,r).$$

The map F' is well defined by Equation (21). We equip $S\times(-r,r)$ with the flat metric induced by the local diffeomorphism F', which extends the metric already defined on $S\times(0,r)$ by identification with W. Since

$$dF'(x,0)(X,T)=dF(x)X+TdF(x)N(x)$$

the metric restricted to $S\times\{0\}$ is the product metric, so the normal to $S\times\{0\}$ in $S\times(-r,r)$ is $N(x,0)=(0,1)$. Since F' is a local isometry, we have for $(x,s)\in S\times(-r,r)$

$$F'\left(\exp_{(x,0)}sN(x,0)\right)=F'(x,0)+sdF'(x,0)(0,1)=F(x)+sdF(x)N(x)=F'(x,s)$$

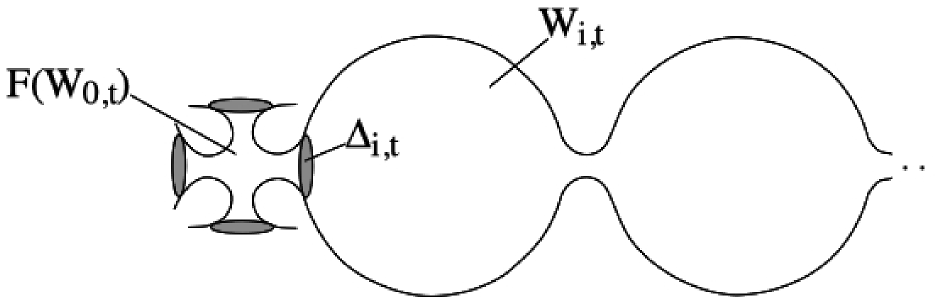

FIGURE 2. Decomposition of a 4-noid into pieces. Only one Delaunay end is represented, and $F(W_{0,t})$ is represented as an embedded domain for clarity, but in general it will be immersed.

Hence $\exp_{(x,0)} sN(x,0) = (x,s)$ so $\mathrm{Tub}_r(S \times \{0\})$ is embedded in $S \times (-r,r)$. □

We now return to the proof of Theorem 1. We orient the minimal n-noid M_0 so that its Gauss map points to the inside in a neighborhood of p_1. For $0 < |t| < \epsilon$, we denote M_t the image of the immersion f_t that we have constructed.

Proposition 7. *If M_0 is Alexandrov embedded, then for $t > 0$ small enough, M_t is Alexandrov embedded.*

Proof: our strategy is to cut M_t by suitable planes into pieces which are either close to M_0 or Delaunay surfaces (see Figure 2). Then we prove that each piece, together with flat disks in the cutting planes, is the boundary of a domain, using the Jordan Brouwer Theorem.

Since M_0 is Alexandrov embedded, N_0 points to the inside in a neighborhood of each end, so M_t has embedded ends by Proposition 6. Let $\varepsilon > 0$ be the number given by our application of Theorem 5 in Section 7.2 and $f_{i,t}^{\mathcal{D}} : \mathbb{C}\backslash\{p_{i,t}\} \to \mathbb{R}^3$ be the Delaunay immersion which approximates f_t in $D^*(p_{i,t},\varepsilon)$. Recall that $f_t(D^*(p_{i,t},\varepsilon))$ is embedded. Let $\widetilde{f}_t = \frac{1}{t} f_t$. By Proposition 5, $\widetilde{f}_t$ converges to ψ on compact subsets of Σ_0, where $\psi : \Sigma_0 \to \mathbb{R}^3$ is a parametrization of M_0. Since M_0 has catenoidal ends, we may assume (taking ε smaller if necessary) that $\psi(D^*(p_{i,0},\varepsilon))$ is embedded and $N_0 \neq N_0(p_{i,0})$ in $D^*(p_{i,0},\varepsilon)$.

Let $h_i : \mathbb{R}^3 \to \mathbb{R}$ be the height function in the direction $N_0(p_{i,0})$, defined by

$$h_i(x) = \langle x, N_0(p_{i,0}) \rangle.$$

We shall cut M_t by the plane $h_i = \delta$ where $\delta > 0$ is a fixed, large enough number such that for $1 \le i \le n$,

$$\delta > \max_{C(p_{i,0},\varepsilon)} h_i \circ \psi.$$

Since $\lim_{z \to p_{i,0}} h_i \circ \psi(z) = +\infty$, we may fix a positive, small enough $\varepsilon' < \varepsilon$ such that

$$\min_{C(p_{i,0},\varepsilon')} h_i \circ \psi > \delta.$$

Let $\mathcal{A}_{i,t}$ be the annulus defined by $\varepsilon' \le |z - p_{i,t}| \le \varepsilon$. Since $N_0 \neq N_0(p_{i,0})$ in $\mathcal{A}_{i,0}$,

$$\min_{\mathcal{A}_{i,0}} \|N_0(z) - N_0(p_{i,0})\| > 0.$$

For $t>0$ small enough:

$$\max_{C(p_{i,t},\varepsilon)} h_i\circ\widetilde{f}_t<\delta \tag{22}$$

$$\min_{C(p_{i,t},\varepsilon')} h_i\circ\widetilde{f}_t>\delta \tag{23}$$

$$\min_{\mathcal{A}_{i,t}} \|N_t(z)-N_0(p_{i,0})\|>0. \tag{24}$$

Hence the function $h_i\circ\widetilde{f}_t$ has no critical point in the annulus $\mathcal{A}_{i,t}$. So $h_i\circ\widetilde{f}_t=\delta$ defines a regular closed curve $\gamma_{i,t}$ in $\mathcal{A}_{i,t}$. At $t=0$, $h_i\circ\psi=\delta$ is a single curve around $p_{i,0}$, so $\gamma_{i,t}$ has only one component and is not contractible in $\mathcal{A}_{i,t}$. Let $D_{i,t}\subset\mathbb{C}$ be the topological disk bounded by $\gamma_{i,t}$ and $D^*_{i,t}=D_{i,t}\setminus\{p_{i,t}\}$. Let $\Delta_{i,t}$ be the closed topological disk bounded by $\widetilde{f}_t(\gamma_{i,t})$ in the plane defined by $h_i(x)=\delta$.

Claim 2. *For $t>0$ small enough, $\widetilde{f}_t(D^*_{i,t})\cap\Delta_{i,t}=\emptyset$.*

Proof: of course, $h_i\circ\widetilde{f}_t>\delta$ in $D^*_{i,t}\cap\mathcal{A}_{i,t}$. What we need to prove is that $\widetilde{f}_t(D^*(p_{i,t},\varepsilon'))$ does not intersect $\Delta_{i,t}$. We do this by comparison with the Delaunay surface. Let $\Pi_i=N_0(p_{i,0})^\perp$ and $\pi_i=\mathbb{R}^3\to\Pi_i$ be the orthogonal projection. Since ψ has a catenoidal end at $p_{i,0}$, $\psi(\mathcal{A}_{i,t})$ is a graph over an annulus in the plane Π_i, with inside boundary circle $\pi_i\circ\psi(C(p_{i,t},\varepsilon))$ and outside boundary circle $\pi_i\circ\psi(C(p_{i,t},\varepsilon'))$. Moreover, N_0 is close to $N_0(p_{i,t})$. Since $\widetilde{f}_t$ is C^1 close to ψ in $\mathcal{A}_{i,t}$, for $t>0$ small enough, $\widetilde{f}_t(\mathcal{A}_{i,t})$ is a graph over an annulus in the plane Π_i, with inside boundary circle $\pi_i\circ\widetilde{f}_t(C(p_{i,t},\varepsilon))$ and outside boundary circle $\pi_i\circ\widetilde{f}_t(C(p_{i,t},\varepsilon'))$.

Now we go back to the original scale. Since f_t is C^1 close to $f^{\mathcal{D}}_{i,t}$ in $D^*(p_{i,t},\varepsilon)$, we conclude that $f^{\mathcal{D}}_{i,t}(\mathcal{A}_{i,t})$ is a graph over an annulus in the plane Π_i, with inside boundary circle $\pi_i\circ f^{\mathcal{D}}_{i,t}(C(p_{i,t},\varepsilon))$ and outside boundary circle $\pi_i\circ f^{\mathcal{D}}_{i,t}(C(p_{i,t},\varepsilon'))$. Then from the geometry of Delaunay surfaces, there exists a curve $\gamma_{i,t,0}$ in $D^*(p_{i,t},\varepsilon')$ such that $f^{\mathcal{D}}_{i,t}(\gamma_{i,t,0})$ is a closed curve in the plane $h_i=\frac{1}{2}$. Let $D_{i,t,0}$ be the disk bounded by $\gamma_{i,t,0}$ and $\mathcal{A}_{i,t,0}$ be the closed annulus bounded by $\gamma_{i,t}$ and $\gamma_{i,t,0}$. Then $h_i\circ f^{\mathcal{D}}_{i,t}>\frac{1}{2}$ in $D_{i,t,0}$ and $f^{\mathcal{D}}_{i,t}(\mathcal{A}_{i,t,0})$ is a graph over an annulus in the plane Π_i. Since f_t is C^1 close to $f^{\mathcal{D}}_{i,t}$ in $D^*(p_{i,t},\varepsilon)$, we conclude that $h_i\circ f_t>\frac{1}{4}$ in $D^*_{i,t,0}$ and $f_t(\mathcal{A}_{i,t,0})$ is a graph over an annulus in the plane Π_i.

Back to the scale $\frac{1}{t}$, $\widetilde{f}_t(D_{i,t}\cap\mathcal{A}_{i,t,0})$ is a graph over an annulus in the plane Π_i whose inside boundary circle is $\pi_i\circ\widetilde{f}_t(\gamma_{i,t})=\pi_i(\partial\Delta_{i,t})$, so $\widetilde{f}_t(D_{i,t}\cap\mathcal{A}_{i,t,0})\cap\Delta_{i,t}=\emptyset$. Moreover, $h_i\circ\widetilde{f}_t>\frac{1}{4t}\gg\delta$ in $D^*_{i,t,0}$ so $\widetilde{f}_t(D^*_{i,t,0})\cap\Delta_{i,t}=\emptyset$. □

Claim 3. *For $t>0$ small enough, $\widetilde{f}_t(D^*_{i,t})\cup\Delta_{i,t}$ is the boundary of a cylindrically bounded domain $W_{i,t}\subset\mathbb{R}^3$.*

Proof: since f_t is close to $f^{\mathcal{D}}_{i,t}$ in $D^*_{i,t}$, we can find an increasing diverging sequence $(R_{t,k})_{k\in\mathbb{N}}$ such that $f_t(D^*_{i,t})$ intersects the plane $h_i=R_{t,k}$ transversally along a closed curve $f_t(\gamma_{i,t,k})$. (Explicitely, we can take $R_{t,k}=\frac{1}{2}+k\,h_i(T_t)$ where $T_t\in\mathbb{R}^3$ is the period of the Delaunay surface $f^{\mathcal{D}}_{i,t}$.) Let $\mathcal{A}_{i,t,k}$ be the annulus bounded by $\gamma_{i,t}$ and $\gamma_{i,t,k}$. Let $\Delta_{i,t,k}$ be the closed disk bounded by $\widetilde{f}_t(\gamma_{i,t,k})$ in the plane $h_i = t^{-1}R_{t,k}$. Then $\widetilde{f}_t(\mathcal{A}_{i,t,k}) \cup \Delta_{i,t} \cup \Delta_{i,t,k}$ is topologically a sphere: the image of $\mathbb{S}^2$ by an injective continuous map. By the Jordan Brouwer Theorem, it is the boundary of a bounded domain $W_{i,t,k}$. Clearly, $W_{i,t,k} \subset W_{i,t,k+1}$. We take $W_{i,t} = \bigcup_{k\in\mathbb{N}} W_{i,t,k}$. □

Let $\Omega_t = \mathbb{C} \cup \{\infty\} \setminus (D_{1,t} \cup \cdots \cup D_{n,t})$. Let W' be the flat 3-manifold given by Lemma 1 and denote $F : W' \to \mathbb{R}^3$ its developing map (instead of F'). (Here W' is an open manifold, meaning not a manifold-with-boundary.)

Claim 4. *For $t > 0$ small enough, there exists a compact domain $W_{0,t}$ in W' such that*

$$F(\partial W_{0,t}) = \widetilde{f}_t(\Omega_t) \cup \Delta_{1,t} \cup \cdots \cup \Delta_{n,t}.$$

Proof: by definition, ψ lifts to a diffeomorphism $\widehat{\psi} : \Sigma_0 \to S \subset W'$ such that $F \circ \widehat{\psi} = \psi$. Since M_0 has catenoidal ends, there exists domains $V_1, \cdots, V_n$ in W' such that for $1 \le i \le n$:

- $F : V_i \to F(V_i) \subset \mathbb{R}^3$ is a diffeomorphism,
- V_i is foliated by flat disks on which $h_i \circ F$ is constant (in particular, $h_i \circ F$ is constant on ∂V_i),
- $\widehat{\psi}(D^*(p_{i,0}, \varepsilon)) \subset V_i$ (which might require taking a smaller $\varepsilon > 0$),
- $h_i < \delta$ on $V_i \cap \widehat{\psi}(\Sigma_0 \setminus \bigcup_{i=1}^n D(p_{i,0}, \varepsilon))$ (which might require taking a larger δ).

Let $r > 0$ be the radius of the embedded tubular neighborhood of S in W' constructed in Lemma 1. For $t > 0$ small enough, $||\widetilde{f}_t - \psi|| < r$ in $\overline{\Omega}_t$, so $\widetilde{f}_t$ lifts to $\widehat{f}_t : \overline{\Omega}_t \to W'$ such that $F \circ \widehat{f}_t = \widetilde{f}_t$. (Explicitely, $\widehat{f}_t(z) = \exp_{\widehat{\psi}(z)}(\widetilde{f}_t(z) - \psi(z))$.) From the properties of V_i and the convergence of $\widehat{f}_t$ to $\widehat{\psi}$ on compact subsets of Σ_0, we have for $t > 0$ small enough

$$\widehat{f}_t(\overline{\Omega}_t \cap D(p_{i,0}, \varepsilon)) \subset V_i \tag{25}$$

$$h_i < \delta \quad \text{on } V_i \cap \widehat{f}_t(\Sigma_0 \setminus \textstyle\bigcup_{i=1}^n D(p_{i,0}, \varepsilon)). \tag{26}$$

By Equation (25), $\widehat{f}_t(\gamma_{i,t}) \subset V_i$ so $\Delta_{i,t}$ lifts to a closed disk $\widehat{\Delta}_{i,t} \subset V_i$ such that $\partial\widehat{\Delta}_{i,t} = \widehat{f}_t(\gamma_{i,t})$ and $F(\widehat{\Delta}_{i,t}) = \Delta_{i,t}$. Since F is a diffeomorphism on V_i, $\widehat{f}_t(\Omega_t \cap D(p_{i,t}, \varepsilon))$ is disjoint from $\widehat{\Delta}_{i,t}$. By (26), $\widehat{f}_t(\Omega_t \setminus \bigcup_{i=1}^n D(p_{i,t}, \varepsilon))$ is disjoint from $\widehat{\Delta}_{i,t}$. Hence $\widehat{f}_t(\Omega_t) \cap \widehat{\Delta}_{i,t} = \emptyset$. Then $\widehat{f}_t(\Omega_t) \cup \widehat{\Delta}_{1,t} \cup \cdots \cup \widehat{\Delta}_{n,t}$ is a topological sphere in W'. Since M_0 has genus zero, W' is homeomorphic to $\mathbb{R}^3$. By the Jordan Brouwer Theorem, $\widehat{f}_t(\Omega_t) \cup \widehat{\Delta}_{1,t} \cup \cdots \cup \widehat{\Delta}_{n,t}$ is the boundary of a compact domain $W_{0,t} \subset W'$. □

Returning to the proof of Proposition 7, let W_t be the abstract 3-manifold with boundary obtained as the disjoint union $\overline{W}_{0,t} \sqcup \overline{W}_{1,t} \sqcup \cdots \sqcup \overline{W}_{n,t}$, identifying $\overline{W}_{0,t}$ and $\overline{W}_{i,t}$ along their boundaries $\widehat{\Delta}_{i,t}$ and $\Delta_{i,t}$ via the map F for $1 \le i \le n$. Let $F_t : W_t \to \mathbb{R}^3$ be the map defined by $F_t = F$ in $\overline{W}_{0,t}$ and $F_t = \mathrm{id}$ in $\overline{W}_{i,t}$ for $1 \le i \le n$. Then F_t is a proper local diffeomorphism whose boundary restriction parametrizes M_t. Moreover, since each $\overline{W}_{i,t}$ is homeomorphic to a closed ball minus a boundary point, we may compactify W_t by adding n points. This proves that M_t is Alexandrov-embedded. □

Appendix A. Appendix: On Delaunay ends in the DPW method

As already said, Delaunay ends in the DPW method have been studied in [13], where it is proved that if ξ is a holomorphic perturbation of the standard Delaunay potential, the resulting immersion has a Delaunay end at $z = 0$ and is close to a Delaunay surface in a disk $D(0, \varepsilon)$. In case the potential ξ_t depends on a parameter t and the Delaunay residue has weight $\sim t$, their result has been refined by T. Raujouan in [15], yielding a uniform ε as $t \to 0$. This is delicate because the corresponding Fuchsian system is resonant at $t = 0$. The result of [15] is the key to proving embeddedness of the CMC n-noids constructed in [18].

We consider the standard Delaunay residue for $t \leq \frac{1}{16}$:

$$A_t(\lambda) = \begin{pmatrix} 0 & \lambda^{-1}r + s \\ \lambda r + s & 0 \end{pmatrix} \qquad \text{where } \begin{cases} r + s = \frac{1}{2} \\ rs = t \\ r < s \end{cases}$$

In particular, in the limit case $t = 0$, we have

$$A_0 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Definition 6 ([15])**.** *A perturbed Delaunay potential is a family of DPW potentials ξ_t of the form*

$$\xi_t(z, \lambda) = A_t(\lambda)\frac{dz}{z} + R_t(z, \lambda)dz$$

where R_t is of class C^2 with respect to $(t, z, \lambda) \in (-T, T) \times D(0, \varepsilon) \times \mathbb{A}_\rho$ for some positive ε and T, and satisfies $R_0 = 0$. In particular, $\xi_0 = A_0\frac{dz}{z}$.

Let (e_1, e_2, e_3) represent the canonical basis of $\mathbb{R}^3$ in the $\mathfrak{su}(2)$ model.

Theorem 5. *Let ξ_t be a perturbed Delaunay potential. Let $\Phi_t(z, \lambda)$ be a family of solutions of $d\Phi_t = \Phi_t\xi_t$ in the universal cover of the punctured disk $D^*(0, \varepsilon)$. Assume that $\Phi_t(z, \lambda)$ depends continuously on (t, z, λ) and that the Monodromy Problem for Φ_t is solved. Let $f_t = \mathrm{Sym}(\mathrm{Uni}(\Phi_t))$ be the immersion given by the DPW method. Finally, assume that $\Phi_0(1, \cdot)$ is constant (i.e. independent of λ).*

Given $0 < \alpha < 1$, there exists uniform positive numbers $\varepsilon' \leq \varepsilon$, $T' \leq T$, c and a family of Delaunay immersions $f_t^{\mathcal{D}} : \mathbb{C}^* \to \mathbb{R}^3$ such that:

(1) For $0 < |t| < T'$ and $0 < |z| < \varepsilon'$:

$$\|f_t(z) - f_t^{\mathcal{D}}(z)\| \leq c|t|\,|z|^\alpha.$$

(2) For $0 < t < T'$, $f_t : D^*(0, \varepsilon') \to \mathbb{R}^3$ is an embedding.

(3) The end of $f_t^{\mathcal{D}}$ at $z = 0$ has weight $8\pi t$ and its axis converges when $t \to 0$ to the half-line spanned by the vector Qe_3Q^{-1} where

$$Q = \mathrm{Uni}\left(\Phi_0(1)H\right) \qquad \text{and} \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Thomas Raujouan has proved this result in [15], Theorem 3, in the case $\Phi_0(1, \lambda) = I_2$. He proves that the limit axis is spanned by e_1. (In fact, he finds that the limit axis is $-e_1$, but this is because he has the opposite sign in the Sym-Bobenko formula. See Remark 3.) Then in Section 2 of [15], he explains, in the case $r > s$, how to extend his result to the case where $\Phi_0(1, \lambda)$ is constant. We adapt his method to the case $r < s$.

Lemma 2. *Under the assumptions of Theorem 5, there exists a gauge $G(z)$ and a change of variable $h(z)$ with $h(0)=0$ such that $\widetilde{\xi}_t=(h^*\xi_t)\cdot G$ is a perturbed Delaunay potential (with residue A_t) and $\widetilde{\Phi}_t=(h^*\Phi_t)\times G$ satisfies at $t=0$*

$$\widetilde{\Phi}_0(1,\lambda)=QH^{-1}\in\Lambda SU(2). \tag{27}$$

Proof: we follow the method explained in Section 2 of [15]. We take the change of variable in the form

$$h(z)=\frac{z}{pz+q}$$

where p,q are complex numbers (independent of t) to be determined, with $q\neq 0$. We consider the following gauge:

$$G(z)=\frac{1}{\sqrt{q(pz+q)}}\begin{pmatrix} pz+q & pz\\ 0 & q\end{pmatrix}.$$

It is chosen so that

$$G(0)=I_2 \quad\text{and}\quad dG=GA_0\frac{dz}{z}-A_0G\frac{dh}{h}. \tag{28}$$

(In fact, the gauge G is found as the only solution of Problem (28) which is upper triangular.) We have

$$\widetilde{\xi}_t=(h^*\xi_t)\cdot G=G^{-1}\left(A_t(\lambda)\frac{dh}{h}+R_t(h,\lambda)dh\right)G+G^{-1}dG.$$

Since $G(0)=I_2$, $\widetilde{\xi}_t$ has a simple pole at $z=0$ with residue A_t. Using Equation (28), we obtain at $t=0$:

$$\widetilde{\xi}_0=G^{-1}A_0\frac{dh}{h}G+G^{-1}dG=A_0\frac{dz}{z}.$$

Hence $\widetilde{\xi}_t$ is a perturbed Delaunay potential. It remains to compute $\widetilde{\Phi}_0(1)$. The matrix H diagonalises A_0:

$$A_0=H\begin{pmatrix}\frac{-1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}H^{-1}$$

Hence

$$\Phi_0(z)=\Phi_0(1)z^{A_0}=\Phi_0(1)H\begin{pmatrix}\frac{1}{\sqrt{z}} & 0\\ 0 & \sqrt{z}\end{pmatrix}H^{-1}$$

$$\begin{aligned}
\widetilde{\Phi}_0(1) &= \Phi_0(h(1))G(1)\\
&= \Phi_0(1)H\begin{pmatrix}\sqrt{p+q} & 0\\ 0 & \frac{1}{\sqrt{p+q}}\end{pmatrix}H^{-1}\frac{1}{\sqrt{q(p+q)}}\begin{pmatrix}p+q & p\\ 0 & q\end{pmatrix}\\
&= \Phi_0(1)H\begin{pmatrix}\sqrt{q} & \frac{p}{\sqrt{q}}\\ 0 & \frac{1}{\sqrt{q}}\end{pmatrix}H^{-1}
\end{aligned}$$

We decompose $\Phi_0(1)H=QR$ with $Q\in SU(2)$ and $R=\begin{pmatrix}\rho & \mu\\ 0 & \frac{1}{\rho}\end{pmatrix}$. Then

$$\widetilde{\Phi}_0(1)=Q\begin{pmatrix}\rho & \mu\\ 0 & \frac{1}{\rho}\end{pmatrix}\begin{pmatrix}\sqrt{q} & \frac{p}{\sqrt{q}}\\ 0 & \frac{1}{\sqrt{q}}\end{pmatrix}H^{-1}$$

We take $q = \frac{1}{\rho^2}$ and $p = -\frac{\mu}{\rho}$ to cancel the two matrices in the middle and obtain Equation (27). $\square$

We can now prove Theorem 5. Let

$$\widehat{\Phi}_t(z,\lambda) = HQ^{-1}\widetilde{\Phi}_t(z,\lambda) = HQ^{-1}\Phi_t(h(z),\lambda)G(z,\lambda)$$

Since $\widehat{\Phi}_0(1,\lambda) = I_2$, we can apply Theorem 3 in [15] which says that the resulting immersion $\widehat{f}_t$ satisfies Points 1 and 2 of Theorem 5 and its limit axis is spanned by e_1. We have

$$f_t \circ h = QH^{-1}\widehat{f}_t HQ^{-1}$$

so $f_t \circ h$ and $\widehat{f}_t$ differ by a rotation and the limit axis of f_t is spanned by the vector $QH^{-1}e_1HQ^{-1}$. Now

$$H^{-1}e_1H = -\frac{\mathrm{i}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = -\mathrm{i}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = e_3.$$

$\square$

References

[1] D. Brander, J. Dorfmeister: *Deformations of constant mean curvature surfaces preserving symmetries and the Hopf differential.* Ann. Sc. Norm. Super. Pisa Cl. Sci. vol. XIV, issue 2, (2015), 645–675.

[2] C. Cosín, A. Ros: *A Plateau problem at infinity for properly immersed minimal surfaces with finite total curvature.* Indiana Univ. Math. J. 50, no. 2 (2001), 847–879.

[3] J. Dorfmeister, F. Pedit, H. Wu: *Weierstrass type representation of harmonic maps into symmetric spaces.* Communications in Analysis and Geometry 6 (1998), 633-668.

[4] J. Dorfmeister, H. Wu: *Construction of constant mean curvature n-noids from holomorphic potentials.* Mathematische Zeitschrift 258 (2008), 773–803.

[5] O. Forster: *Lectures on Riemann surfaces.* Graduate texts in Mathematics, Springer Verlag (1981).

[6] S. Fujimori, S. Kobayashi, W. Rossman: *Loop group methods for constant mean curvature surfaces.* arXiv:math/0602570.

[7] I. Gohberg: *A factorization problem in normed rings, functions of isometric and symmetric operators and singular integral equations.* Russian Math. Survey 19, No 1 (1964), 63–144

[8] K. Gröchenig: *Weight functions in time-frequency analysis.* Fields Institute Communications 52, (2007), 343–366.

[9] K. Große-Brauckmann, R. Kusner, J. Sullivan: *Triunduloids: embedded constant mean curvature surfaces with three ends and genus zero.* J. Reine Angew. Math. 564 (2003), 35–61.

[10] K. Große-Brauckmann, R. Kusner, J. Sullivan: *Coplanar constant mean curvature surfaces.* Comm. Anal. Geom. 15 (2007), no. 5, 985–1023.

[11] N. Kapouleas: *Complete constant mean curvature surfaces in euclidean three-space.* Annals of Mathematics 131 (1990), 239–330.

[12] M. Kilian: *Constant mean curvature cylinders.* Doctoral thesis, Univ. of Massachusetts, (2000).

[13] M. Kilian, W. Rossman, N. Schmitt: *Delaunay ends of constant mean curvature surfaces.* Compositio Mathematica 144 (2008), 186–220.

[14] R. Mazzeo, F. Pacard: *Constant mean curvature surfaces with Delaunay ends.* Comm. Anal. and Geom. 9, no. 1 (2001), 169–237.

[15] T. Raujouan: *On Delaunay ends in the DPW method.* To appear in Indiana Univ. Math. J. arXiv:1710.00768v2 (2017).

[16] N. Schmitt: *Constant mean curvature trinoids.* arXiv:math/0403036 (2004).

[17] N. Schmitt, M. Kilian, S. Kobayashi, W. Rossman: *Unitarization of monodromy representations and constant mean curvature trinoids in 3-dimensional space forms.* Journal of the London Mathematical Society 75 (2007), 563–581.

[18] M. Traizet: *Construction of constant mean curvature n-noids using the DPW method.* To appear in Journal für die reine und angewandte Mathematik. arXiv 1709.00924 (2017).

[19] M. Traizet: *Opening nodes in the DPW method.* arXiv 1808.01366 (2018).