The classification of singly periodic minimal surfaces with genus zero and Scherk-type ends

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Abstract. Given an integer $k \geq 2$, let $\mathcal{S}(k)$ be the space of complete embedded singly periodic minimal surfaces in $\mathbb{R}^3$, which in the quotient have genus zero and $2k$ Scherk-type ends. It is well-known that $\mathcal{S}(2)$ consists of the 1-parameter family of singly periodic Scherk minimal surfaces. We prove that for each $k \geq 3$, there exists a natural one-to-one correspondence between $\mathcal{S}(k)$ and the space of convex unitary nonspecial polygons through the map which assigns to each $M \in \mathcal{S}(k)$ the polygon whose edges are the flux vectors at the ends of M (a special polygon is a parallelogram with two sides of length 1 and two sides of length $k-1$). As consequence, $\mathcal{S}(k)$ reduces to the saddle towers constructed by Karcher [5].

1 Introduction.

In 1834, Scherk [17] discovered a singly periodic, embedded minimal surface in euclidean space $\mathbb{R}^3$. This surface, known as *Scherk's second surface,* has four ends asymptotic to vertical half planes. Geometrically it may be seen as the desingularization of two perpendicular vertical planes. In 1988, Karcher [5] proposed a 1-parameter deformation of Scherk's second surface, which may be seen as the desingularization of two vertical planes with an angle θ between them (θ is the parameter). This deformation is now called the family of *Scherk singly periodic minimal surfaces.* They all have four ends asymptotic to vertical half planes (these are called *Scherk-type ends*) and genus zero in the quotient $\mathbb{R}^3/T$ by the shortest orientation-preserving translation T. In the same paper, Karcher constructed for each integer $k \geq 3$ a $(2k-3)$-parameter family of embedded singly periodic

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minimal surfaces which have $2k$ Scherk-type ends and genus zero in the quotient by the period. These surfaces are now called *Karcher saddle towers.*

The main result of this paper is

Theorem 1 *Let M be a properly embedded singly periodic minimal surface in $\mathbb{R}^3$ with a finite number of Scherk-type ends and genus zero in the quotient by the period. Then M is a singly periodic Scherk surface or a Karcher saddle tower.*

An interesting problem is to classify all properly embedded minimal surfaces in flat 3-manifolds, which have genus zero and finite topology (namely, a finite number of ends). The above theorem allows us to complete the classification in the case of the flat manifold $\mathbb{R}^2 \times \mathbb{S}^1 = \mathbb{R}^3/T$. Indeed, by a theorem of Meeks and Rosenberg [11], in this case all ends must be simultaneously asymptotic to horizontal planes, helicoids or vertical halfplanes (Scherk-type ends). The first case is impossible when M has genus zero by the maximum principle for minimal surfaces. In the second case, the only example is the helicoid by a theorem of Pérez and Ros [13]. Therefore we obtain

Corollary 1 *Let M be a non flat properly embedded minimal surface in $\mathbb{R}^2 \times \mathbb{S}^1$. Assume M has genus zero and finite topology. Then M is a helicoid, a singly periodic Scherk surface, or a Karcher saddle tower.*

The classification of the properly embedded minimal planar domains with finitely many ends is also complete in the following ambient spaces: In $\mathbb{R}^3$, M must be a plane, a catenoid (Collin [1], López and Ros [7]) or a helicoid (Meeks and Rosenberg [10]). In $\mathbb{T}^2 \times \mathbb{R}$, M must be a doubly periodic Scherk surface (Lazard-Holly and Meeks [6]). In $\mathbb{T}^3$ there are no examples since M must have genus at least 3 (Meeks [8]). The only case that remains open is the flat manifold $\mathbb{R}^3/S_\theta$ where S_θ is a screw motion.

The proof of theorem 1 is a modified application of the machinery developed by Meeks, Pérez and Ros in their characterization of Riemann minimal examples [9]. It is known [11] that M must have an even number $2k \geq 4$ of ends (this is a simple consequence of embeddedness). For $k \geq 2$, let $\mathcal{S}(k)$ be the space of properly embedded singly periodic minimal surfaces with genus zero and $2k$ Scherk-type ends. In the case $k = 2$, it is elementary to see that $\mathcal{S}(2)$ reduces to the family of Scherk singly periodic surfaces. When $k \geq 3$, the goal is to prove that $\mathcal{S}(k)$ reduces to the space $\mathcal{K}(k)$ of Karcher saddle towers with $2k$ ends. The argument is based on modeling $\mathcal{S}(k)$ as an analytical subset in a complex manifold $\mathcal{W}(k)$ of finite dimension (roughly, $\mathcal{W}(k)$ consists of all admissible Weierstrass data for our problem). Then the procedure has three steps:

- *Properness:* Uniform curvature estimates are proven for a sequence of surfaces in $\mathcal{S}(k)$ provided one has some control on the *flux* at the ends.
- *Openness:* Any surface in $\mathcal{S}(k) \setminus \mathcal{K}(k)$ may be deformed into another surface in $\mathcal{S}(k) \setminus \mathcal{K}(k)$ by perturbing locally the flux at the ends. Together with the properness property, this implies that if $\mathcal{S}(k) \setminus \mathcal{K}(k)$ is non empty, then any configuration of flux may be achieved by surfaces in $\mathcal{S}(k) \setminus \mathcal{K}(k)$.
- *Local uniqueness:* We prove that some particular configuration of fluxes may only be achieved by surfaces in $\mathcal{K}(k)$. This proves that $\mathcal{S}(k) \setminus \mathcal{K}(k)$ is empty.

The paper is organized as follows. In Section 2 we recall the necessary background for our problem. Furthermore, we provide an elementary proof of the case $k = 2$, we remind the construction of the Karcher saddle towers and we define the *flux map,* which is the main tool to prove our main theorem. In Section 3 we study the space of admissible Weierstrass data. In Section 4 we prove our main theorem assuming the properness, openness and local uniqueness statements. Properness is studied in Section 5 and openness in Section 6. In Sections 7 and 8 we prove local uniqueness in a neighborhood of some limit cases.

2 Preliminaries.

Let $\widetilde{M} \subset \mathbb{R}^3$ be a properly embedded minimal surface invariant by the translation of vector $T = (0, 0, 2\pi)$. $\widetilde{M}$ induces a properly embedded minimal surface $M = \widetilde{M}/T \subset \mathbb{R}^3/T = \mathbb{R}^2 \times \mathbb{S}^1$. Meeks and Rosenberg [11] proved that if M has finite topology, then it has finite total curvature and so, M is conformally a finitely punctured closed Riemann surface. Furthermore, M has an even number of ends, all of them simultaneously asymptotic to nonvertical planes, vertical helicoids or flat vertical annuli. These asymptotic behaviors are called respectively *planar, helicoidal* or *Scherk-type ends.* From now on we assume that M has genus zero and Scherk-type ends.

For a fixed integer $k \geq 2$, we will denote by $\mathcal{S}(k)$ the space of properly embedded singly periodic minimal surfaces which are invariant by the translation $T = (0, 0, 2\pi)$, with genus zero in the quotient and $2k$ Scherk-type ends, modulo translations and rotations around the x_3-axis. $\mathcal{S}(k)$ can be naturally endowed with the uniform topology on compact sets of $\mathbb{R}^3/T$. Any surface $M \in \mathcal{S}(k)$ is conformally equivalent to the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ minus $2k$ points $p_1, \ldots, p_{2k}$ that correspond to the ends of M. The Gauss map (stereographically projected) g of the singly periodic lifting $\widetilde{M} \subset \mathbb{R}^3$ of M descends to the quotient surface, giving rise to a meromorphic map on M that extends holomorphically through each puncture (with $|g(p_j)| = 1$ because the ends are asymptotic to flat vertical annuli). The degree of such an extension $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is given by the Meeks-Rosenberg formula [11], which in this setting writes

$$\deg(g) = k - 1. \tag{1}$$

The *height differential* $\phi = \frac{\partial x_3}{\partial z} dz$, where x_3 is the third coordinate function on $\widetilde{M}$ and z is a local conformal coordinate, also descends to a meromorphic differential on M which extends through each p_j having a simple pole with residue $\pm i$ (this comes from the normalization of the period vector to be $\pm T$). Hence the flux vector F_j at the end p_j, defined as the integral of the inner unit conormal vector to M along the boundary of an end representative of p_j, is given by

$$F_j = 2\pi u_j, \tag{2}$$

where u_j is a unitary horizontal vector in $\mathbb{R}^3$ (in fact, u_j is orthogonal to the limit normal vector at p_j). The Divergence Theorem implies that

$$\sum_{j=1}^{2k} u_j = 0. \tag{3}$$

For the remainder of the paper, we will use the identification $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ given by $(a, b, c) \equiv (a + ib, c)$. Hence, we can write $u_j = e^{i\theta_j} \in \mathbb{S}^1 = \{|z| = 1\} \subset \mathbb{C}$ with $\theta_i \in \mathbb{R}$.

The goal of this paper is to classify all elements in $\mathcal{S}(k)$ for each $k \geq 3$. Although the case $k = 2$ is well-known, we include an elementary proof of the description of $\mathcal{S}(2)$ for the sake of completeness.

Lemma 1 *$\mathcal{S}(2)$ consists of the 1-parameter family of singly periodic Scherk surfaces.*

Proof. Fix $M \in \mathcal{S}(2)$, with ends p_1, p_2, p_3, p_4 cyclically ordered. Using (2) and (3), it is easy to check that $F_1 = -F_3$ and $F_2 = -F_4$. This implies that, up to a rotation around the x_3-axis, the values at the ends of the stereographically projected extended Gauss map

g of M are $g(p_1) = e^{i\theta}$, $g(p_2) = e^{-i\theta}$, $g(p_3) = -g(p_1)$ and $g(p_4) = -g(p_2)$ for certain $\theta \in (0, \pi/2)$. Since equation (1) says that $\deg(g) = 1$, we can parametrize M by its Gauss map, i.e. $M = \overline{\mathbb{C}} \setminus \{\pm e^{\pm i\theta}\}$ with $g(z) = z$. As the height differential ϕ of M has simple poles at $\pm e^{\pm i\theta}$ and simple zeros at $0, \infty$ (because $0, \infty$ are finite points of M), we have

$$\phi = c \frac{z\, dz}{\prod(z \pm e^{\pm i\theta})},$$

where $c \in \mathbb{C}^* = \mathbb{C} - \{0\}$. Finally, as $\operatorname{Res}_{p_j} \phi = \pm i$ for each j we deduce that $c = 4\sin(2\theta)$, hence the Weierstrass data (g, ϕ) represent a singly periodic Scherk minimal surface. $\square$

For the remainder of the paper we will only deal with $2k$-ended surfaces, $k \geq 3$. Given $M \in \mathcal{S}(k)$ with ends $p_1, \ldots, p_{2k}$, there is a natural cyclic ordering on the set of ends so that the arguments θ_j of the flux vectors $F_j = 2\pi e^{i\theta_j}$ satisfy

$$\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{2k} \leq \theta_1 + 2\pi. \tag{4}$$

but there is no natural choice for the first end p_1. The surface M together with the choice of the first end p_1 is what we will call a *marked surface* (unless it leads to confusion, we will keep the notation M for a marked surface). Let $\widetilde{\mathcal{S}}(k)$ be the set of such marked minimal surfaces, which naturally inherits the uniform topology on compact sets. Since our surfaces are defined up to rotations about the x_3-axis, we can assume that $F_1 = 1$ for each $M \in \widetilde{\mathcal{S}}(k)$.

Let $\mathcal{U}(k)$ be the set of *marked convex* $2k$*-gons* in $\mathbb{C}$ with edges of unit length. By marked we mean that for each polygon we choose a vertex. To avoid translations and rotations, we normalize so that the points $z = 0$ and $z = 1$ are consecutive vertices of each element in $\mathcal{U}(k)$ and the chosen vertex is 0. We will identify $\mathcal{U}(k)$ with the space

$$\left\{ \mathbf{u} = (u_1, \ldots u_{2k}) = (1, e^{i\theta_2}, \cdots, e^{i\theta_{2k}}) \in (\mathbb{S}^1)^{2k} \mid \text{ equations (3) and (4) hold } \right\},$$

so that a list $\mathbf{u}$ corresponds to the marked polygon $P_{\mathbf{u}}$ of consecutive vertices $Z_1 = 0$, $Z_j = Z_{j-1} + e^{i\theta_j}$ for $j = 1, \ldots, 2k$. $\mathcal{U}(k)$ can be seen as a subset of the space $\mathcal{V}(k)$ of unitary $2k$-gons, not necessarily convex or embedded. $\mathcal{V}(k)$ has a natural structure of real analytic manifold of dimension $2k - 3$ (with the angles at consecutive vertices as local parameters). Since polygons in $\mathcal{U}(k)$ are convex but not necessarily strictly convex, $\mathcal{U}(k)$ is a closed subset of $\mathcal{V}(k)$. Clearly that $\mathcal{U}(k)$ is connected, and its boundary consists of those convex $2k$-gons with at least one edge of length 2 (viewed as two consecutive edges).

Definition 1 An element $\mathbf{u} \in \mathcal{U}(k)$ is said to be a *special polygon* if there exist $v, w \in \mathbb{S}^1$ so that two edges of $P_\mathbf{u}$ are equal to $\pm v$ and all other edges are equal to $\pm w$. In other words, $P_\mathbf{u}$ is a (possibly degenerated) parallelogram with two edges of length 1 and two edges of length $k-1$, these last ones considered as $k-1$ consecutive unitary edges of $P_\mathbf{u}$. We will denote by $\mathcal{U}_0(k)$ the subset of special polygons. The limit case $v = \pm w$ represents a parallelogram that degenerates in the segment with end points $0, k$. We will call $\mathbf{u}_0(k)$ this degenerate parallelogram.

Definition 2 With the notation above, we define the *flux map* $F : \widetilde{\mathcal{S}}(k) \to \mathcal{U}(k)$ as $M \in \widetilde{\mathcal{S}}(k) \mapsto F(M) = \mathbf{u}$, where $\mathbf{u} = (u_1, \ldots, u_{2k})$ is defined by (2). Since the u_j are the flux vectors at the ends of M (up to $\frac{1}{2\pi}$), F is clearly continuous. A marked surface $M \in \widetilde{\mathcal{S}}(k)$ is called *special* if its flux polygon $F(M)$ is special. We will denote by $\widetilde{\mathcal{S}}_0(k) = F^{-1}(\mathcal{U}_0(k))$ to the space of special marked surfaces.

Remark 1 *A simple consequence of the maximum principle for minimal surfaces insures that* $F^{-1}(\mathbf{u}_0(k)) = \emptyset$.

2.1 Karcher saddle towers.

We now recall how to construct a surface $M \in \widetilde{\mathcal{S}}(k)$ from a given nonspecial $2k$-gon (for details, see Karcher [5]). Given $\mathbf{u} \in \mathcal{U}(k) \setminus \mathcal{U}_0(k)$, a theorem by Jenkins-Serrin [4] insures that there exists a minimal graph $G_\mathbf{u}$ with boundary values alternately $+\infty$ and $-\infty$ on the sides of the polygon $P_\mathbf{u}$ (which we see inside the plane $\{z = 0\} \subset \mathbb{R}^3$). Since $G_\mathbf{u}$ is bounded by vertical lines over the vertices of $P_\mathbf{u}$, the conjugate surface $G^*_\mathbf{u}$ of $G_\mathbf{u}$ is bounded by horizontal arcs which lie alternately in two horizontal planes. $G^*_\mathbf{u}$ can be extended by reflection in such horizontal arcs to a surface $M_\mathbf{u} \in \mathcal{S}(k)$ called a *Karcher saddle tower,* whose flux polygon is $\mathbf{u}$. Since $\mathcal{U}(k)$ has $2k-3$ freedom parameters, we deduce that the Karcher saddle towers come in a $(2k-3)$-parameter family $\mathcal{K}(k)$ of examples.

Since each surface $M_\mathbf{u} \in \mathcal{K}(k)$ has genus zero and admits a reflective symmetry R_Π across a horizontal plane Π such that the set of fixed points of R_Π coincides with $M_\mathbf{u} \cap \Pi$, Theorem 1 in Cosín and Ros [2] insures that the only bounded Jacobi functions on $M_\mathbf{u}$ are linear functions of its Gauss map. This condition and an Implicit Function Theorem argument (see for instance Pérez and Ros [14]) imply that $\mathcal{K}(k)$ is open in $\mathcal{S}(k)$. By construction, $\mathcal{K}(k)$ is also closed in $\mathcal{S}(k)$. The continuous dependence on the $2k$-gon $\mathbf{u}$ of the Jenkins-Serrin graph $G_\mathbf{u}$ gives that the map $\mathbf{u} \in \mathcal{U}(k) \setminus \mathcal{U}_0(k) \mapsto M_\mathbf{u} \in \widetilde{\mathcal{S}}(k)$ is continuous. Since $\mathcal{U}(k) \setminus \mathcal{U}_0(k)$ is connected, we deduce that $\mathcal{K}(k)$ forms a component of $\mathcal{S}(k)$. Similarly as with other spaces of surfaces, $\widetilde{\mathcal{K}}(k)$ will stand for the space of marked Karcher saddle towers, which is open and closed in $\widetilde{\mathcal{S}}(k)$.

Remark 2 *The hypothesis of the Jenkins-Serrin theorem is as follows: for each strict sub-polygon Q of $P_{\mathbf{u}}$ (which means that the vertices of Q form a strict subset of those of $P_{\mathbf{u}}$), the perimeter of Q must be strictly larger than twice the number of edges of $Q \cap P_{\mathbf{u}}$ marked with $+\infty$ (resp. $-\infty$). While this hypothesis is satisfied by any polygon in $\mathcal{U}(k) \setminus \mathcal{U}_0(k)$, it fails to be true for special polygons: simply consider any sub-rhombus of an $\mathbf{u} \in \mathcal{U}_0(k)$ (we thank Barbara Nelli for pointing out this fact to us).*

Another consequence of the continuity of the map $\mathbf{u} \in \mathcal{U}(k) \setminus \mathcal{U}_0(k) \mapsto M_{\mathbf{u}} \in \widetilde{\mathcal{S}}(k)$ is that we can state equivalently Lemma 1 by saying that $F : \widetilde{\mathcal{S}}(2) \to \mathcal{U}(2) \setminus \{\mathbf{u}_0(2)\}$ is a homeomorphism. Our main theorem now states as follows.

Theorem 2 *Assume $k \geq 3$. Then $\mathcal{S}(k) = \mathcal{K}(k)$ and the map $F : \widetilde{\mathcal{S}}(k) \to \mathcal{U}(k) \setminus \mathcal{U}_0(k)$ is a homeomorphism, whose inverse map is $u \mapsto M_{\mathbf{u}}$ (in particular, $\widetilde{\mathcal{S}}_0(k) = \emptyset$).*

Since each Karcher saddle tower $M_{\mathbf{u}}$ is obtained by symmetrization of the conjugate graph $G^*_{\mathbf{u}}$ across the horizontal planes containing $\partial G^*_{\mathbf{u}}$, we obtain directly the following corollary.

Corollary 2 *Any properly embedded singly periodic minimal surface with genus zero and Scherk-type ends has an horizontal plane of symmetry.*

3 The space of Weierstrass representations.

Let $M \in \widetilde{\mathcal{S}}(k)$ be an embedded marked surface with ends $p_1, \ldots, p_{2k}$, flux polygon $F(M) = \mathbf{u} = (u_1, \ldots, u_{2k})$, complex Gauss map g and height differential ϕ. The limit normal values are $g(p_j) = \pm i u_j$, $1 \leq j \leq 2k$. Since M is embedded, the normal vector always points to the same component of $(\mathbb{R}^3/T) \setminus M$, hence we may assume without loss of generality that

$$g(p_j) = (-1)^{j+1} i u_j, \tag{5}$$

and thus the flux at the end p_j is given in terms of (g, ϕ) by

$$F_j = 2\pi u_j = -2\pi g(p_j) \operatorname{Res}_{p_j} \phi = 2\pi (-1)^j i u_j \operatorname{Res}_{p_j} \phi,$$

which yields

$$\operatorname{Res}_{p_j}\phi = (-1)^j i, \qquad 1 \leq j \leq 2k. \tag{6}$$

Let $\mathcal{W}(k)$ be the set of lists $(g, p_1, \dots, p_{2k})$ where g is a degree $k-1$ meromorphic function on $\overline{\mathbb{C}}$ and $p_1, \dots, p_{2k} \in \mathbb{C}^*$ are $2k$ distinct points. Elements in $\mathcal{W}(k)$ will be simply denoted by g. By using Hurwitz schemes, one can naturally endow $\mathcal{W}(k)$ with a structure of a complex analytic manifold of dimension $4k-4$ (the symmetric polynomials on the $2k-4$ branch values of each $g \in \mathcal{W}(k)$ together with the $2k$ points $p_j \in \mathbb{C}^*$ give a local chart for $\mathcal{W}(k)$). Given an element $g \in \mathcal{W}(k)$, we define the *height differential* ϕ associated to g as the unique meromorphic 1-form on $\overline{\mathbb{C}}$ with polar divisor

$$(\phi)_\infty = \prod_{j=1}^{2k} p_j,$$

whose residues are given by equation (6). Let $\mathcal{M}(k)$ be the set of elements $g \in \mathcal{W}(k)$ such that (g, ϕ) is the Weierstrass pair of a complete immersed minimal surface in $\mathbb{R}^3/T$ with Scherk-type ends, or equivalently

$$\begin{aligned} \mathcal{M}(k) &= \{g \in \mathcal{W}(k) \mid (\phi)_0 = (g)_0 (g)_\infty \text{ and } |g(p_j)| = 1,\ 1 \leq j \leq 2k\} \\ &= \{g \in \mathcal{W}(k) \mid (\phi)_0 = (g)_0 (g)_\infty \text{ and } |\operatorname{Res}_{p_j}(g\phi)| = 1,\ 1 \leq j \leq 2k\}. \end{aligned}$$

(Here $(h)_0, (h)_\infty$ denote the zero and polar divisors of a meromorphic function or differential on $\overline{\mathbb{C}}$). We can see the set of marked embedded surfaces $\widetilde{\mathcal{S}}(k)$ as a subset of $\mathcal{M}(k)$. Note that that $\widetilde{\mathcal{S}}(k)$ is not open in $\mathcal{M}(k)$ (because the directions of the ends is not fixed; compare with the situation in Meeks-Pérez-Ros [9] or Pérez-Rodríguez-Traizet [12]).

Remark 3 *Essentially what we do in the paper is prove that the only elements of $\mathcal{M}(k)$ which give* embedded *surfaces are the Karcher saddle towers. We will do this using a rather elaborate machinery developed in previous papers [9]. As elements of $\mathcal{M}(k)$ are defined by simple algebraic equations, one may wonder if there is a purely algebraic proof (following, for example, the method in Wei [19]). So let us explain why a purely computational proof can not succeed.*

We have used embeddedness to obtain algebraic restrictions on the Weierstrass data (such as equation (5) above). However, there are examples of singly periodic minimal surfaces with Scherk-type ends and genus zero which satisfy all conditions we have written *so far and yet are not embedded. Here is one example:*

$$g = \frac{z^4 - 4iz^2 + 3}{z^4 + 4iz^2 + 3}, \qquad \phi = \frac{(z^4 - 4iz^2 + 3)(z^4 + 4iz^2 + 3)i\,dz}{(z^2+3)(z^2-3)(z^2+1)(z^2-1)z}.$$

This example has 10 ends. The value of the complex Gauss map at the ends $0, \infty$ *is* 1*, while the ends at* $\pm 1, \pm i, \pm\sqrt{3}, \pm i\sqrt{3}$ *have Gauss map* $\pm i$*. This is an example of a* special *surface, and it is not embedded. Hence embeddedness must be used in a rather strong way, which is hard to perform in a purely computational proof.*

Definition 3 The *flux map* $F : \mathcal{W}(k) \to \mathbb{C}^{2k}$ is given by

$$F(g) = (\,\mathrm{Res}_{\,p_1}(g\phi), \ldots, \mathrm{Res}_{\,p_{2k}}(g\phi))\,,$$

which is clearly holomorphic. Given $\mathbf{u} \in \mathcal{U}(k)$, let $\mathcal{M}_{\mathbf{u}}(k) = F^{-1}(\mathbf{u}) \cap \mathcal{M}(k)$ and $\widetilde{\mathcal{S}}_{\mathbf{u}}(k) = F^{-1}(\mathbf{u}) \cap \widetilde{\mathcal{S}}(k)$ (note that $F^{-1}(\mathbf{u})$ is not necessarily contained in $\mathcal{M}(k)$, since ϕ might not satisfy the regularity condition of the induced metric: $(\phi)_0 = (g)_0(g)_\infty$)

Lemma 2 *Given* $\mathbf{u} \in \mathcal{U}(k)$*, the subset* $\widetilde{\mathcal{S}}_{\mathbf{u}}(k)$ *is open and closed in* $\mathcal{M}_{\mathbf{u}}(k)$*.*

Proof. Closeness follows since a limit of embedded surfaces is itself embedded. Any embedded surface in $\widetilde{\mathcal{S}}_{\mathbf{u}}(k)$ admits a regular neighborhood of constant positive radius (this is a consequence of the maximum principle if $\mathbf{u}$ is in the interior of $\mathcal{U}(k)$, and of the maximum principle at infinity if $\mathbf{u} \in \partial\mathcal{U}(k)$, see Ros [16] for a similar argument). From here the desired openness is standard. □

We will need later the following property.

Lemma 3 *Compact analytic subvarieties of* $\mathcal{W}(k)$ *are finite sets.*

Proof. (Inspired in Meeks-Pérez-Ros [9]). We may assume after composition with Möbius transformations that for any element $g \in \mathcal{W}(k)$ we have $p_1 = \infty, p_2 = 0, p_3 = 1$. Then any element in $\mathcal{W}(k)$ has $k-1$ zeros and $k-1$ poles in $\mathbb{C}^* - \{1\}$. Consider the elementary symmetric functions $\sigma_1, \ldots, \sigma_{k-1}$ of the zeros, which are holomorphic functions on $\mathcal{W}$. Let A be a compact analytic subvariety of $\mathcal{W}(k)$. As $\sigma_j(A)$ is a compact analytic subvariety of $\mathbb{C}$, $\sigma_j(A)$ is finite. Since this works for any j, we deduce that there are only a finite number of possibilities in $\mathbb{C}^* - \{1\}$ for zeros of elements of A. In the same way, there are only a finite number of possibilities for poles of elements of A. Finally, the function $g \in \mathcal{W}(k) \mapsto g(p_1)$ is holomorphic, so again there are only a finite number of possibilities for $g(p_1)$, $g \in A$. As any $g \in \mathcal{W}$ is locally determined by its zeros, poles and its value in $\mathbb{C}^*$ at one point, this proves that A is discrete. As A is compact, it must be finite. $\square$

3.1 The ligature map.

The ligature map L will be a holomorphic mapping from $\mathcal{W}(k)$ to $\mathbb{C}^{4k-4}$ that expresses when an element $g \in \mathcal{W}(k)$ lies in $\mathcal{M}(k)$. To define L we cannot simply consider the value of ϕ/dz at the zeros and poles of g, because g might have multiple zeros and in this case these zeros do not depend analytically on g. We may again assume by normalization that $p_1 = \infty$, $p_2 = 0$ and $p_3 = 1$ and consequently write

$$g = \lambda \frac{P_1(z)}{P_2(z)}, \qquad \phi = i \frac{P_3(z)}{\prod_{j \geq 2}(z - p_j)} dz,$$

where P_1, P_2 and P_3 are unitary polynomials of respective degrees $k-1$, $k-1$ and $2k-2$ (this last degree comes from the fact that ϕ must have a simple pole at ∞; the factor i in front of $\frac{P_3}{\prod_{j \geq 2}(z-p_j)} dz$ comes from the equation $\mathrm{Res}_{\,\infty}\phi = -i$, which follows from (6)). Note that the polynomials P_1, P_2, P_3 depend analytically on g. On the other hand, the regularity condition $(\phi)_0 = (g)_0 (g)_\infty$ is equivalent to $P_1 P_2 = P_3$, or also to

$$\mathrm{Remainder}\left(\frac{P_3}{P_1 P_2}\right) = 0.$$

As the above remainder of the euclidean division of P_3 by $P_1 P_2$ is a complex polynomial of degree less than $2k-2$, we can see it as a tuple in $\mathbb{C}^{2k-2}$ by considering its coefficients.

Definition 4 The *ligature map* $L : \mathcal{W}(k) \to \mathbb{C}^{4k-4}$ is the holomorphic map

$$g \in \mathcal{W}(k) \mapsto L(g) = \left(\mathrm{Remainder}(P_3/(P_1 P_2)),\, \mathrm{Res}_{\,p_3}(g\phi), \ldots, \mathrm{Res}_{\,p_{2k}}(g\phi)\right).$$

We only consider the above residues for $3 \leq j \leq 2k$ since the equations to solve (expressing when $g \in \mathcal{W}(k)$ lies in $\mathcal{M}(k)$) are not independent. Indeed, by the Residue Theorem

$$\sum_{\text{zeros of } g} \mathrm{Res}\,(g^{-1}\phi) + \sum_{j=1}^{2k} \mathrm{Res}_{\,p_j}(g^{-1}\phi) = 0,$$

$$\sum_{\text{poles of } g} \operatorname{Res}(g\phi) + \sum_{j=1}^{2k} \operatorname{Res}_{p_j}(g\phi) = 0.$$

Also observe that

$$\operatorname{Res}_{p_j}(g^{-1}\phi) = \frac{(-1)^j i}{g(p_j)} = \frac{-1}{\operatorname{Res}_{p_j} g\phi}.$$

Thus if $L(g) = (0, \overset{(2k-2)}{\ldots}, 0, u_3, \ldots, u_{2k})$ with $\mathbf{u} = (u_1, \ldots, u_k) \in \mathcal{U}(k)$, then we obtain a system of two equations for $(\operatorname{Res}_{p_1}(g\phi), \operatorname{Res}_{p_2}(g\phi))$, whose two solutions are (u_1, u_2) and (u_2, u_1). Hence $\mathcal{M}_{\mathbf{u}}(k)$ is a union of components of the subset of $\mathcal{W}(k)$ defined by $L(g) = (0, u_3, \ldots, u_{2k})$. Since L is holomorphic, we deduce the following lemma.

Lemma 4 *Given $\mathbf{u} \in \mathcal{U}(k)$, $\mathcal{M}_{\mathbf{u}}(k)$ is a complex analytic subvariety of $\mathcal{W}(k)$.*

4 Proof of Theorem 2.

The proof of Theorem 2 is by induction on $k \geq 3$. Suppose that the theorem holds for any $k' < k$, and we will prove it for k (note that the theorem holds for $k = 2$). We will assume the following three propositions, to be proven in further sections. Recall that $\widetilde{\mathcal{S}}_0(k) = F^{-1}(\mathcal{U}_0(k))$ is the set of *special* surfaces with $2k$ ends.

Proposition 1 (Properness)
(i) The flux map $F : \widetilde{\mathcal{S}}(k) \setminus \widetilde{\mathcal{S}}_0(k) \to \mathcal{U}(k) \setminus \mathcal{U}_0(k)$ is proper.
(ii) The flux map $F : \widetilde{\mathcal{S}}_0(k) \to \mathcal{U}_0(k)$ is proper.

Proposition 2 (Openness) *The flux map $F : \widetilde{\mathcal{S}}(k) \to \mathcal{U}(k)$ is open.*

Proposition 3 (Local uniqueness) *There exists a point $\mathbf{u}_* \in \mathcal{U}_0(k)$ and $\varepsilon > 0$ such that if $\mathbf{u} \in \mathcal{U}(k)$ satisfies $\|\mathbf{u} - \mathbf{u}_*\| < \varepsilon$, then $F^{-1}(\mathbf{u}) \subset \widetilde{\mathcal{K}}(k)$.*

Assuming these results, we now prove Theorem 2. We first check that $\widetilde{\mathcal{S}}_0(k)$ is empty. By Proposition 2 and an elementary topological argument, the flux map $F : \widetilde{\mathcal{S}}_0(k) \to \mathcal{U}_0(k)$ is open. The same map is proper by point *(ii)* of Proposition 1. Hence the image by F of any component of $\widetilde{\mathcal{S}}_0(k)$ is an entire component of $\mathcal{U}_0(k)$. Note that $\mathcal{U}_0(k)$ is not connected but in any component of $\mathcal{U}_0(k)$ there is a degenerate polygon which reduces to a segment. By the maximum principle (see Remark 1), such a polygon cannot lie in the image of F. Therefore $\widetilde{\mathcal{S}}_0(k) = \emptyset$.

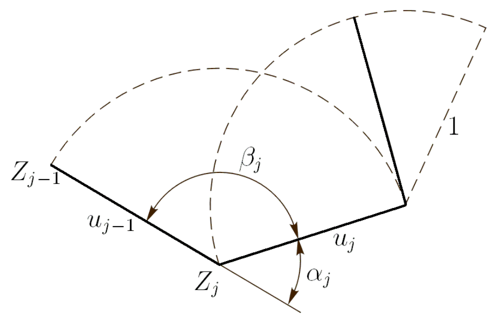

Figure 1: The angles α_j, β_j at the vertex Z_j between u_{j-1} and u_j.

Then by Proposition 2 and elementary topology, the flux map $F : \widetilde{\mathcal{S}}(k) \to \mathcal{U}(k) \setminus \mathcal{U}_0(k)$ is open. By point *(i)* of Proposition 1, it is also proper. Hence the image by F of any component of $\widetilde{\mathcal{S}}(k)$ is all of $\mathcal{U}(k) \setminus \mathcal{U}_0(k)$ (which is connected). By Proposition 3, any component of $\widetilde{\mathcal{S}}(k)$ must contain some Karcher saddle towers. Since the set of saddle towers $\widetilde{\mathcal{K}}(k)$ form a component of $\widetilde{\mathcal{S}}(k)$, we deduce that $\widetilde{\mathcal{S}}(k) = \widetilde{\mathcal{K}}(k)$. Now the remaining assertions in the statement of Theorem 2 follow easily.

Remark 4 *Although we did not use the hypothesis of induction in the last paragraph, the proof of Proposition 1 needs Theorem 2 to be true for any $k' < k$.*

5 Properness.

Along this section, $\{M_n\}_n \subset \widetilde{\mathcal{S}}(k)$ will denote a sequence of marked surfaces whose associated flux polygons $\mathbf{u}_n = F(M_n)$ converge as $n \to \infty$ to a polygon $\mathbf{u}_\infty \in \mathcal{U}(k)$. The goal of this section is to understand the limit of (a subsequence of) $\{M_n\}_n$.

5.1 Preliminaries on convex polygons.

We need an elementary fact about convex unitary polygons. Given $\mathbf{u} = (1, u_2, \ldots, u_{2k}) \in \mathcal{U}(k)$ and $j \in 1, \ldots, 2k$, let α_j be the angle between u_{j-1} and u_j (here we convine $u_0 = u_{2k}$). Thus $\beta_j = \pi - \alpha_j$ is the inner angle of the polygon $P_\mathbf{u} \subset \mathbb{C}$ associated to $\mathbf{u}$ at the vertex between the edges $\sum_{i=1}^{j-1} u_k$ and $\sum_{i=1}^{j} u_j$ of $P_\mathbf{u}$, see Figure 1.

Lemma 5 *Given $\mathbf{u} \in \mathcal{U}(k)$ and $j = 1 \ldots, 2k$, it holds $\alpha_j + \alpha_{j+1} \le \pi$, with equality for some j if and only if $\mathbf{u} \in \mathcal{U}_0(k)$.*

Proof. Fix $j = 1, \dots 2k$. We divide the polygon $P_{\mathbf{u}}$ in two edges and two chains of consecutive edges in the following way.

- We will call L_1 to the edge of $P_{\mathbf{u}}$ whose extreme points are the vertices Z_{j-1}, Z_j of $P_{\mathbf{u}}$ with respective inner angles β_{j-1}, β_j.
- Starting at Z_{j-1}, let $L_1', L_2', \dots, L_{k-1}'$ be a chain of $k-1$ consecutive edges of $P_{\mathbf{u}}$, so that $L_1 \cap L_1' = \{Z_{j-1}\}$ and each couple L_h', L_{h+1}' share a common end point.
- Similarly, let $L_1'', L_2'', \dots, L_{k-1}''$ be a chain of $k-1$ consecutive edges of $P_{\mathbf{u}}$ starting at Z_j, such that $L_1 \cap L_1'' = \{Z_j\}$.
- We will call L_k to the remaining edge of $P_{\mathbf{u}}$, so that the two above chains are joined in $P_{\mathbf{u}}$ at their respective end points by L_1 and L_k.

Let $\Omega \subset \mathbb{C}$ be the closed halfstrip bounded by L_1, the halfline starting at Z_{j-1} that contains L_1' and its parallel halfline starting at Z_j. Lemma 5 can be equivalently stated as follows.

Assertion 1 *If $P_{\mathbf{u}} \subset \Omega$, then $\mathbf{u} \in \mathcal{U}_0(k)$.*

Proof of Assertion 1. Firstly note that for $k = 2$ the assertion is trivial, so assume $k > 2$. We divide the convex hull of $P_{\mathbf{u}}$ into $k-1$ parallelograms $Q_1, \dots, Q_{k-1}$ so that

(A) Q_1 is bounded by L_1, L_1', L_1'' and $L_1'' - L_1$.

(B) Q_2 is constructed as Q_1 exchanging L_1 by $L_2 = L_1'' - L_1'$, L_1' by L_2' and L_1'' by L_2''.

(C) Q_h is constructed inductively from Q_{h-1} as in item **(B)**.

Note that all the subpolygons Q_i contain two edges of $P_{\mathbf{u}}$, except Q_1 and Q_{k-1} which have three. Using that Q_1 has three unitary edges and lies in Ω, an elementary trigonometry argument shows that $|L_2| \leq 1$ (here $|L_2|$ means length), with equality if and only if $L_1'' \subset \partial\Omega$. In particular, we can assume $|L_2| < 1$ and argue by contradiction (otherwise Q_1 is a rhombus so we eliminate Q_1 from $P_{\mathbf{u}}$ and argue with the remaining polygon). Since $|L_2| < 1$, the edges L_1', L_1'' extend to bigger segments which form the boundary of a triangle T_1, and $P_u \subset T_1$ by convexity. Now consider the second subpolygon Q_2. The situation is similar to the one for L_1 (here we use that the corresponding triangle T_2 lies inside T_1), with the only difference that the basis L_2 of Q_2 is not unitary but $|L_2| < 1$. We claim that still $|L_3| < 1$ holds true: For this, enlarge L_2 to have length 1 keeping the same L_2' and exchange L_2'' by a translated copy of it. Clearly the length of the new L_3 is bigger than the length of the former L_3 (otherwise we contradict convexity for Q_2). But the enlarged subpolygon Q_2 lies in the situation of the first case with respect to the halfstrip between L_2' and $L_2 + L_2'$, hence the new L_3 has length less than or equal to 1, and so the former L_3 satisfies $|L_3| < 1$. Repeating the process, we arrive at $|L_h| < 1$ for all h. But the last L_h is $L_k \subset P_{\mathbf{u}}$, which is unitary, a contradiction. □

Remark 5 *Clearly Lemma 5 does not hold for convex unitary polygons with an odd number of vertices.*

5.2 Area, flux and curvature estimates.

As usual, we denote by $\widetilde{M}$ the lift to $\mathbb{R}^3$ of a surface $M \subset \mathbb{R}^3/T$, by $B(x, R)$ the open ball centered at $x \in \mathbb{R}^3$ with radius $R > 0$, by $C(R) \subset \mathbb{R}^3/T$ the solid vertical cylinder of radius R and axis $\mathbb{R}T$ modded out by T, and by K_Σ the Gauss curvature function of a surface $\Sigma \subset \mathbb{R}^3$.

Lemma 6 *Given $M \in \mathcal{M}(k)$, the area of $\widetilde{M} \cap B(R)$ is less than $k\pi R^2$ for any $R > 0$.*

Proof. Since $\widetilde{M}$ is asymptotic to $2k$ vertical halfplanes, the limit of $\frac{\text{Area}(\widetilde{M} \cap B(R))}{R^2}$ when $R \to \infty$ is $k\pi$. Now the result follows from the monotonicity formula. □

The fact that the genus of our surfaces is zero allows us to control the flux of vertical sections of our sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}}(k)$ with $\{\mathbf{u}_n = F(M_n)\}_n \to \mathbf{u}_\infty \in \mathcal{U}(k)$.

Lemma 7 *There exist $a \in \mathbb{S}^1$ and $\varepsilon > 0$ depending only on $\mathbf{u}_\infty$, such that for any flat vertical annulus $\Pi \subset \mathbb{R}^3/T$, $n \in \mathbb{N}$ and any component γ_n of $M_n \cap \Pi$,*

$$|\text{Flux}(\gamma_n) \pm 2\pi a| \geq \varepsilon. \tag{7}$$

Proof. Fix a flat vertical annulus $\Pi \subset \mathbb{R}^3/T$ and a component γ_n of $M_n \cap \Pi$. Since M_n has genus zero, γ_n separates M_n in two components. Hence the flux of M_n along γ_n is equal to the sum of the fluxes at some of the ends of M_n. In particular, $\text{Flux}(\gamma_n)$ is horizontal (we will see it in $\mathbb{C}$) and for n fixed there are only a finite number of values in $\mathbb{C}$ for $\text{Flux}(\gamma_n)$, that only depend on $\mathbf{u}_n$. Now the existence of $a \in \mathbb{S}^1$ and $\varepsilon > 0$ verifying (7) follows from the fact that $\{\mathbf{u}_n\}_n$ converges to $\mathbf{u}_\infty$ as $n \to \infty$. $\square$

Let g_n be the Gauss map of M_n (stereographically projected from the north pole).

Lemma 8 *Let $a \in \mathbb{S}^1$ be the unit complex number given by Lemma 7. Then, there exists $\delta > 0$ depending only on $\mathbf{u}_\infty$ such that for all $n \in \mathbb{N}$ and $p_n \in M_n$ such that $g_n(p_n) = ia$,*

$$\sup \left| K_{(M_n - p_n) \cap C(10)} \right| \geq \delta.$$

Proof. By contradiction, if $\sup \left| K_{(M_n - p_n) \cap C(10)} \right|$ converges to zero as $n \to \infty$ then $\{M_n - p_n) \cap C(10)\}_n$ converges uniformly to the flat vertical annulus Π_1 with normal ia. Now take a flat vertical annulus Π perpendicular to Π_1. Then, the flux of M_n along $M_n \cap \Pi = \gamma_n$ converges to $\pm 2\pi a$ as $n \to \infty$, which contradicts Lemma 7. $\square$

5.3 Weak limits.

We now prove that $\{M_n\}_n$ converges in some weak sense to a finite number of limit minimal surfaces. This ought to be true in a quite general setup, although we will give a proof enough for our setting.

Let $a \in \mathbb{S}^1$ be given by Lemma 7. For each $n \in \mathbb{N}$ let $p_{1,n}, \ldots, p_{k-1,n} \in M_n$ the points such that $g_n(p_{j,n}) = ia$ for $1 \leq j \leq k-1$ (counting with multiplicity). Let $\widetilde{p}_{1,n}, \ldots, \widetilde{p}_{k-1,n} \in \widetilde{M}_n$ be their corresponding liftings to a fundamental domain of $\widetilde{M}_n$. Fix $j = 1, \ldots, k-1$. Since the degree of g_n is fixed $k-1$, given $R > 0$ there exists $c_1 > 0$ such that the absolute total curvature of $(\widetilde{M}_n - \widetilde{p}_{j,n}) \cap B(R)$ is not greater than c_1 for all $n \in \mathbb{N}$. Furthermore, Lemma 6 implies that there exists $c_2 > 0$ such that $\text{Area}\Big((\widetilde{M}_n - \widetilde{p}_{j,n}) \cap B(R)\Big) \leq c_2$ for all $n \in \mathbb{N}$. In this situation, a standard result (see e.g. Theorem 4.40 in [15]) insures that there exists a discrete set $X_j \subset \mathbb{R}^3$ and a properly embedded minimal surface $\widetilde{M}_{j,\infty} \subset \mathbb{R}^3$ such that up to a subsequence, $\{\widetilde{M}_n - \widetilde{p}_{j,n}\}_n$ converges with finite multiplicity in $\mathbb{R}^3 - X_j$ to $\widetilde{M}_{j,\infty}$. Furthermore, for each point $p \in X_j$ and $\mu > 0$, it holds

$$\limsup_k \int_{(\widetilde{M}_n - \widetilde{p}_{j,n}) \cap B(p,\mu)} |K_n| \geq 4\pi. \tag{8}$$

We now distinguish two possibilities, depending whether $\widetilde{M}_{j,\infty}$ is flat or not.

I. If $\widetilde{M}_{j,\infty}$ is not flat, then by a standard argument (see e.g. Section 4 of [15]) the multiplicity of the convergence of $\{\widetilde{M}_n - \widetilde{p}_{j,n}\}_n$ to $\widetilde{M}_{j,\infty}$ is one and $X_j = \emptyset$. Since each $\widetilde{M}_{j,n}$ is T-periodic, the same holds for $\widetilde{M}_{j,\infty}$. The quotient surface $M_{j,\infty} = \widetilde{M}_{j,\infty}/T$ is a planar domain since all the M_n have genus zero, and its absolute total curvature is at most $4\pi(k-1)$. Therefore all ends of $M_{j,\infty}$ are simultaneously planar, helicoidal or Scherk-type by a theorem of Meeks and Rosenberg [11]. As $M_{j,\infty}$ is an embedded nonflat planar domain, the maximum principle implies that $M_{j,\infty}$ does have planar ends. If the ends of $M_{j,\infty}$ are helicoidal, then it is a helicoid (Pérez and Ros [13]). As T is vertical, such a helicoid must be also vertical. Otherwise, $M_{j,\infty}$ has genus zero and at most $2k$ Scherk-type ends.

II. If $\widetilde{M}_{j,\infty}$ is flat, then $X_j \neq \emptyset$ by Lemma 8. The proof of Theorem 4.40 in [15] implies that there exists a sequence of real numbers $\lambda_n \to +\infty$ such that $\lambda_n(\widetilde{M}_n - \widetilde{p}_{j,n})$ converges to a properly embedded nonflat minimal surface $\widehat{M}_{j,\infty}$. Since the M_n are planar domains, $\widehat{M}_{j,\infty}$ must have genus zero and so, it is a catenoid (López and Ros [7]). As the neck of $\widehat{M}_{j,\infty}$ is a closed curve with nonzero period, it must be the uniform limit of closed curves with nonzero flux in the M_n (rescaled by λ_n). As all of such curves on M_n have horizontal flux, $\widehat{M}_{j,\infty}$ is a horizontal catenoid.

Lemma 9 *In the situation above, suppose that $M_{j,\infty}$ is a limit surface of the type* **I**. *Then, none of the ends of $M_{j,\infty}$ has limit normal vector ia.*

Proof. If the ends of $M_{j,\infty}$ are helicoidal, then the Lemma is trivial. Arguing by contradiction, suppose that $M_{j,\infty}$ has Scherk-type ends (which must be vertical because $\widetilde{M}_{j,\infty}$ is T-invariant), one of which has limit normal vector ia. Thus we can find a flat vertical annulus $\Pi \subset \mathbb{R}^3/T$ orthogonal to a such that the flux of $M_{j,\infty}$ along a certain component $\gamma_{j,\infty}$ of $M_{j,\infty} \cap \Pi$ is $\pm 2\pi a$. $\gamma_{j,\infty}$ is the uniform limit of components γ_n of $M_n \cap \Pi$, which therefore have flux converging to $\pm 2\pi a$, a contradiction with Lemma 7. $\square$

We now need to get rid of duplicate limits.

Lemma 10 *In the situation of Lemma 9, let d_j be the degree of the Gauss map on $M_{j,\infty}$ (thus $d_j \leq k-1$). Then, there are another $d_j - 1$ points $q_{1,n}(j), \ldots, q_{d_j-1}(j) \in \{\widetilde{p}_{1,n}, \ldots, \widetilde{p}_{k-1,n}\} \setminus \{\widetilde{p}_{j,n}\}$ such that for every $h = 1 \ldots, d_j - 1$, a translation of $\widetilde{M}_{j,\infty}$ is also the limit of $\{\widetilde{M}_n - q_{h,n}(j)\}_n$ as $n \to \infty$.*

Proof. By Lemma 9, there exists a compact region $C \subset \mathbb{R}^3/T$ such that for n large, g_n takes the value ia at d_j points in C counting multiplicity (one of these points is the origin). The remaining $d_i - 1$ points correspond to other points $q_{1,n}, \ldots, q_{d_j-1} \in \{\widetilde{p}_{1,n}, \ldots, \widetilde{p}_{k-1,n}\} \setminus \{\widetilde{p}_{j,n}\}$, and clearly $\{\widetilde{M}_n - q_{h,n}\}_n$ converges to $\widetilde{M}_{j,\infty}$ (up to a translation) as $n \to \infty$. $\square$

For the remainder of this section, we forget about the repeated limits as $n \to \infty$ of the $M_n - q_{h,n}(j)$ in Lemma 10, since they are the same as $M_{j,\infty}$. Let r be the number of limit surfaces that remain. After reindexing, we obtain that for each $j = 1, \ldots, r$ the sequence $\{M_n - p_{j,n}\}_n$ converges (up to possibly blowing-up) to $M_{j,\infty}$ which is either singly periodic or a horizontal catenoid. By counting the number of times where the Gauss map takes the value ia we deduce that the degrees of the Gauss maps of all the $M_{j,\infty}$ add up to $k-1$. Furthermore, there exist r disjoint metric balls $B_{j,n}$ in $\mathbb{R}^3/T$ such that $M_n \cap B_{j,n}$ traps all the interesting geometry of the limit $M_{j,\infty}$, and the total curvature of $M_n \cap B_{j,n}$ is arbitrarily close to the total curvature of $M_{j,\infty}$. In particular, the total curvature of $M_n \setminus (B_{1,n} \cup \ldots \cup B_{r,n})$ is arbitrarily small.

Note that for $j = 1, \ldots, r$ fixed, the Gauss map at each end representative of $M_{j,\infty}$ is close to a constant that depends on the end. For n large, let Ω be a component of $M_n \setminus (B_{1,n} \cup \ldots \cup B_{r,n})$. By the open mapping theorem, the Gauss map g_n of M_n has to be close to a constant $c(\Omega)$ on Ω (otherwise $g_n(\Omega)$ would cover almost all the sphere). Hence Ω is an extremely flat graph over a certain plane. Now let Ω be a noncompact component of $M_n \setminus (B_{1,n} \cup \ldots \cup B_{r,n})$. Since the ends of M_n have horizontal limit normal vector, we have $|c(\Omega)| = 1$. As the Gauss map of a helicoid in $\mathbb{R}^3/T$ is vertical at its ends, we deduce that for any ball $B_{j_0,n}$ such that $\partial B_{j_0,n} \cap \partial\Omega \neq \emptyset$, the associated limit surface $M_{j_0,\infty}$ is either a horizontal catenoid or a properly embedded planar domain with vertical Scherk-type ends. Therefore for any component Ω' of $M_n \setminus (B_{1,n} \cup \ldots \cup B_{r,n})$ with $\partial B_{j_0,n} \cap \partial\Omega' \neq \emptyset$, it holds $|c(\Omega')| = 1$. We now repeat inductively this process exchanging Ω by Ω', to conclude finally that all the limit surfaces $M_{j,\infty}$ are horizontal catenoids or properly embedded minimal planar domains with vertical Scherk-type ends. We now summarize what we have proven in the following statement.

Proposition 4 *There exists a collection $M_{1,\infty}, \ldots, M_{r,\infty}$ of minimal surfaces and r sequences of homotheties $\{h_{j,n}\}_n$ satisfying*

1. *Each $M_{j,n}$ is a horizontal catenoid or a properly embedded minimal surface of genus zero in $\mathbb{R}^3/T$ with at most $2k$ Scherk-type ends.*

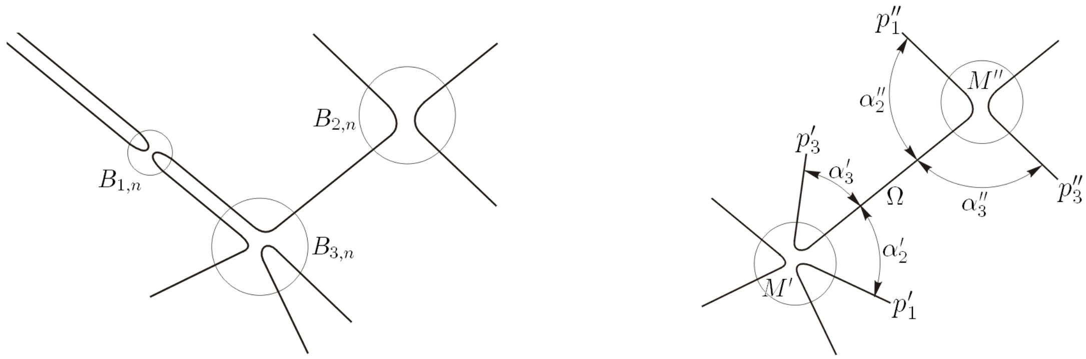

Figure 2: Left: A weak limit configuration with 3 limit surfaces ($M_{1,\infty}$ is a catenoid, $M_{2,\infty}$ a Scherk surface and $M_{3,\infty}$ a surface of genus zero with six Scherk-type ends) and 11 connection pieces (3 of them being compact). Right: See Lemma 11.

2. $\int_{M_n} |K_{M_n}| = \sum_{j=1}^{n} \int_{M_{j,\infty}} |K_{M_{j,\infty}}|$, *for each* $n \in \mathbb{N}$.

3. *If* $M_{j,\infty}$ *is a catenoid, then the sequence of scaling factors of* h_n *blows-up and* $\{h_{j,n}(M_n)\}_n$ *converges smoothly to* $M_{j,\infty}$ *in* $\mathbb{R}^3$.

4. *If* $M_{j,\infty}$ *has Scherk-type ends, then each* h_n *is a translation and* $\{h_{j,n}(M_n)\}_n$ *converges smoothly to* $M_{j,\infty}$ *in* $\mathbb{R}^3/T$.

5. *For* R, n *large, there exist* r *disjoint metric balls* $B_{j,n} \subset \mathbb{R}^3/T$ *such that* $h_{j,n}(B_{j,n})$ *is either the ball* $B(0, R) \subset \mathbb{R}^3$ *(when* $M_{j,\infty}$ *is a catenoid) or the metric ball of radius* R *centered at the origin in* $\mathbb{R}^3/T$ *(when* $M_{j,\infty}$ *has Scherk-type ends), and* M_n *decomposes as*

$$M_n = (M_n \cap B_{1,n}) \cup \ldots \cup (M_n \cap B_{r,n}) \cup \Omega_{1,n} \cup \ldots \cup \Omega_{m,n}.$$

Furthermore, each $\Omega_{h,n}$ *is a graph over a domain in a flat vertical annulus* $\Pi \subset \mathbb{R}^3/T$.

In the sequel, we will call *connection piece* to each of the domains $\Omega_{h,n}$. Note that a connection piece could be compact (if it does not contains ends of M_n), see Figure 2 left.

5.4 Strong compactness.

In this section we shall prove that if $\mathbf{u}_\infty \in \mathcal{U}(k) \setminus \mathcal{U}_0(k)$, then the collection of limit surfaces in Proposition 4 reduces to a single surface in $\widetilde{\mathcal{S}}(k)$ (this is usually referred in literature as *strong compactness* of the original sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}}(k))$, and obtain further information for the noncompact case, i.e. $\mathbf{u}_\infty \in \mathcal{U}_0(k)$.

Assume that $r \geq 2$ (we use the notation in Proposition 4). Consider two limit surfaces $M' = M_{j_1,\infty}$ and $M'' = M_{j_2,\infty}$ which are attached in M_n by a compact connection piece Ω. We label the ends of M' as p'_j and those of M'' as p''_j, in such a way that the ends p'_2 and p''_2 are attached along Ω. Let u'_j and u''_j be the corresponding flux vectors. As in Subsection 5.1, α'_j be the angle between u'_{j-1} and u'_j, and define α''_j analogously with u''_{j-1}, u''_j, see Figure 2 right.

Lemma 11 *In the above situation,* $\alpha'_2 + \alpha''_3 \geq \pi$ *and* $\alpha''_2 + \alpha'_3 \geq \pi$*.*

Proof. By contradiction, suppose $\alpha'_2 + \alpha''_3 < \pi$. If both connection pieces that glue to p'_1, p''_3 are noncompact, then these connection pieces are graphs over flat vertical halfplanes (quotiented by T) hence they intersect, which contradicts that M_n is embedded. Hence at least one of the connection pieces Ω', say that which glues to p'_1, is compact. So p'_1 is attached along Ω' to an end of another weak limit M''' and we repeat the discussion with the same p''_3 by exchanging p'_1 by a suitable end of M'''. After a finite chain of compact connection pieces and weak limits, either p'_1 and p''_3 glue to different ends of M_n, or they glue each other through an almost horizontal compact arc Γ that only intersects the closure of Ω at the end points of Γ. In the first case we find a contradiction as above, while in the second case we join Γ with a suitable almost horizontal arc $\Gamma' \subset \Omega$, so that $\Gamma \cup \Gamma'$ is a closed curve in M_n. Since the metric balls $B_{j,n}$ in Proposition 4 all have bounded radius and for n sufficiently large the length of the parts of $\Gamma \cup \Gamma'$ in the connection pieces is as large as we desire, we deduce that for n large the flux of M_n along $\Gamma \cup \Gamma'$ is not horizontal, which contradicts that such a flux vector is a sum of fluxes at the ends of M_n. Thus $\alpha'_2 + \alpha''_3 \geq \pi$, and similarly $\alpha''_2 + \alpha'_3 \geq \pi$. □

We continue analyzing the case $r \geq 2$. With the notation above, suppose that M', M'' have Scherk-type ends. By Lemma 5 applied to the corresponding polygon fluxes $F(M'), F(M'')$, it holds $\alpha'_2 + \alpha'_3 \leq \pi$ and $\alpha''_2 + \alpha''_3 \leq \pi$. Adding this inequalities and using Lemma 11 we conclude that $\alpha'_2 + \alpha'_3 = \alpha''_2 + \alpha''_3 = \pi$, so Lemma 5 gives that $F(M') \in \mathcal{U}_0(k')$, $F(M'') \in \mathcal{U}_0(k'')$, where k', k'' are half of the number of ends of M', M'' respectively. Furthermore, $F(M')$ and $F(M'')$ have parallel sides. Thus all the fluxes at ends of M' are $\pm 2\pi u'$ except two of them which are $\pm 2\pi v$ (here u, v are distinct points in $\mathbb{S}^1$), and similarly with M''. Note that the connection piece Ω between M' and M'' is attached to ends with flux $\pm 2\pi v$ of M', M'' (otherwise we produce again a closed curve a M_n with nonhorizontal flux vector).

Another consequence of these arguments is that all remaining weak limits $M_{j,\infty}$ other that M', M'' have necessarily Scherk-type ends, so we can repeat the arguments in the last paragraph to obtain that $\mathbf{u}_\infty = \lim F(M_n)$ belongs to $\mathcal{U}_0(k)$. Now we can deduce the main result of this section. Recall that $\mathbf{u}_0(k)$ is the degenerated special polygon all whose edges are parallel (namely, the segment with endpoints $0, k$).

Proposition 5 *Let* $\{M_n\}_n \subset \widetilde{\mathcal{S}}(k)$ *with* $\{F(M_n) = \mathbf{u}_n\}_n \to \mathbf{u}_\infty$.

(A) *if* $\mathbf{u}_\infty \notin \mathcal{U}_0(k)$*, then after passing to a subsequence,* $\{M_n\}_n$ *converges to a single marked surface* $M_\infty \in \widetilde{\mathcal{S}}(k) \setminus \widetilde{\mathcal{S}}_0(k)$.

(B) *If* $\mathbf{u}_\infty \in \mathcal{U}_0(k) \setminus \{\mathbf{u}_0(k)\}$*, then after passing to a subsequence either* $\{M_n\}_n$ *converges to a single marked surface* $M_\infty \in \widetilde{\mathcal{S}}_0(k)$*, or all weak limit surfaces* $M_{j,\infty}$ *are special surfaces with less than* $2k$ *ends.*

(C) *If* $\mathbf{u}_\infty = \mathbf{u}_0(k)$*, then after passing to a subsequence either* $\{M_n\}_n$ *converges to a single marked surface, or all limit surfaces are horizontal catenoids.*

We finish this section by proving Proposition 1. Item *(i)* of this proposition is precisely point **(A)** of Proposition 5. To prove item *(ii)*, consider a sequence of special surfaces $\{M_n\}_n \subset \widetilde{\mathcal{S}}_0(k)$ with $\{F(M_n) = \mathbf{u}_n\}_n \to \mathbf{u}_\infty \in \mathcal{U}_0(k)$.

If $\mathbf{u}_\infty \in \mathcal{U}_0(k) \setminus \{\mathbf{u}_0(k)\}$, then Proposition 5-**(B)** asserts that either the M_n converge to a single marked surface $M_\infty \in \widetilde{\mathcal{S}}_0(k)$ (which is what we want to prove), or all weak limit surfaces $M_{1,\infty}, \dots, M_{r,\infty}$ of the sequence $\{M_n\}_n$ are special surfaces with less than $2k$ ends. By induction hypothesis, Theorem 2 holds for any $k' < k$, so we have $\widetilde{\mathcal{S}}_0(k') = \emptyset$ for each $k' < k$ unless $k' = 2$, hence $M_{1,\infty}, \dots, M_{r,\infty}$ are singly periodic Scherk minimal surfaces. These weak limits are in fact the same Scherk surface, because $\mathbf{u}_\infty \in \mathcal{U}_0(k)$. We will call this limit configuration with $k-1$ copies of a singly periodic Scherk surface a *Scherk limit* (note there exists a 1-parameter family of Scherk limits). In Section 7 we will study each Scherk limit, proving in particular that a sequence of special surfaces cannot converge to a Scherk limit, so this case does not happen.

Finally assume that $\mathbf{u}_\infty = \mathbf{u}_0(k)$. By Proposition 5-**(C)**, either M_n converges to a single marked surface $M_\infty \in \widetilde{\mathcal{S}}_0(k)$ and we are done, or after a blow-up $\{M_n\}_n$ converges to $k-1$ copies of the same horizontal catenoid, a configuration that we will call the *catenoid limit*. In Section 8 we will see that a sequence of special surfaces cannot converge to the catenoid limit, so this case is also impossible. This completes the proof of Proposition 1.

6 Openness.

In this section we will prove Proposition 2. Since openness is a local property, it suffices to see that given $M \in \widetilde{\mathcal{S}}(k)$ there exists an open neighborhood V of M in $\widetilde{\mathcal{S}}(k)$ such that $F|_V : V \to \mathcal{U}(k)$ is an open map. Fix $M \in \widetilde{\mathcal{S}}(k)$. Consider the associated element g in $\mathcal{W}(k)$ and flux polygon $\mathbf{u} = F(M) \in \mathcal{U}(k)$. Clearly $g \in \widetilde{\mathcal{S}}_{\mathbf{u}}(k)$. By Lemma 2 and Lemma 4, $\widetilde{\mathcal{S}}_{\mathbf{u}}(k)$ is an analytic subvariety of $\mathcal{W}(k)$. By the properness Proposition 1, $\widetilde{\mathcal{S}}_{\mathbf{u}}(k)$ is compact, hence it is finite by Lemma 3. Thus there exists an open set $\Omega \subset \mathcal{W}(k)$ such that $g \in \Omega$ and $L^{-1}(L(g)) = \{g\}$. Since L is holomorphic between complex manifolds of the same dimension, the open mapping theorem for finite maps (see [3] page 667) implies there exists an open set $\Omega_1 \subset \Omega$ containing g such that $L|_{\Omega_1}$ is open. Now Proposition 2 follows directly using the relationship between L and F.

7 Uniqueness around any Scherk limit.

In this section we will study surfaces close to the Scherk limits. We will conclude from this study two facts: First, that a sequence of special surfaces cannot converge to a Scherk limit, which was used in the proof of Proposition 1. Second, that all surfaces close to a Scherk limit must be Karcher saddle towers, which will be used to prove Proposition 3.

Recall that a Scherk limit is a configuration of $k-1$ copies of the same singly periodic Scherk minimal surface. In Section 2 we normalized the space $\mathcal{U}(k)$ of convex unitary polygons to have their first component equals 1. In what follows we will change slightly this normalization, which does not affects to the arguments but simplifies the notation. After rotation and suitable choice of the first end for M_n, we may assume that the limit of $\{F(M_n)\}_n$ is $\mathbf{u}_\infty = (\alpha, \overset{(k-1)}{\ldots}, \alpha, 1, -\alpha, \overset{(k-1)}{\ldots}, -\alpha, -1)$ where $\alpha \in \mathbb{S}^1 - \{\pm 1\}$ and the ends $p_1, \ldots, p_{2k}$ are labeled as indicated in Figure 3. A little thought of how the normal vector behaves when gluing consecutive copies of the Scherk surface, shows that if we fix the normal map g to be i at the end p_{2k}, then it alternates the values $\pm i$ in consecutive annular connection pieces between copies of the Scherk surface, finishing at $g(p_k) = (-1)^{k+1} i$.

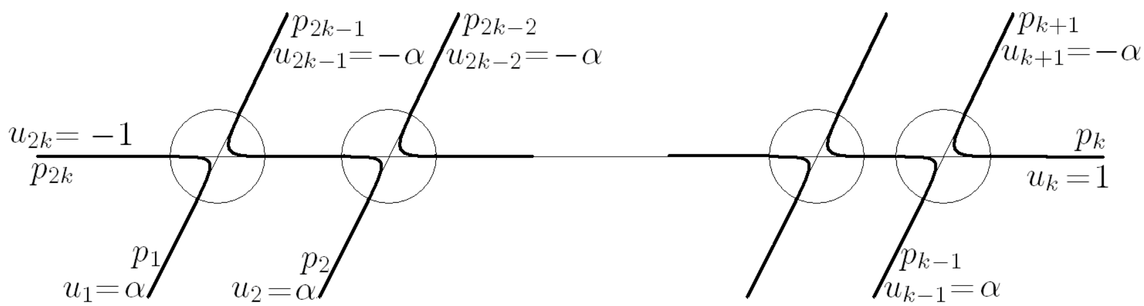

Figure 3: A Scherk limit configuration, with the ends and the corresponding fluxes u_j.

7.1 Weierstrass data.

A model for an element in $\mathcal{W}(k)$ close to the Scherk limit is as follows. Fix $\mathbf{u} = (u_1, \ldots, u_{2k}) \in \mathbb{C}^{2k}$ close to $\mathbf{u}_\infty$ (we do not require the components of $\mathbf{u}$ either to be unitary or to satisfy equation (3)). Consider $k-1$ copies $\overline{\mathbb{C}}_1, \ldots, \overline{\mathbb{C}}_{k-1}$ of $\overline{\mathbb{C}}$. For $1 < j < k$, consider complex numbers $a_j \neq b_j$ close to $(-1)^{j+1} i$. Glue $\overline{\mathbb{C}}_{j-1}$ with $\overline{\mathbb{C}}_j$ along a cut from a_j to b_j in the usual way, so that the chain of $k-1$ copies of $\overline{\mathbb{C}}$ yields a compact surface Σ of genus zero. On Σ we consider the meromorphic function g defined as $g = z$ in each $\overline{\mathbb{C}}_j$. Then g has degree $k-1$ and its branch values are a_j, b_j, $1 < j < k$. This function g together with the ordered collection of points $p_1, \ldots, p_{2k} \in \Sigma$ given below determine an element of $\mathcal{W}(k)$.

- For $1 \leq j < k$, p_j and p_{2k-j} are the unique points in $\overline{\mathbb{C}}_j$ that satisfy $g(p_j) = p_j = (-1)^{j+1} i u_j$, $g(p_{2k-j}) = p_{2k-j} = (-1)^{j+1} i u_{2k-j}$ (compare with equation (5)). Note that since $\mathbf{u}$ is close to $\mathbf{u}_\infty$ and $\alpha \neq \pm 1$, we deduce that p_j, p_{2k-j} are far from the neck between $\overline{\mathbb{C}}_{j-1}$ and $\overline{\mathbb{C}}_j$.

- p_k lies in $\overline{\mathbb{C}}_{k-1}$ and p_{2k} in $\overline{\mathbb{C}}_1$, are determined by $g(p_k) = p_k = (-1)^{k+1} i u_k$, $g(p_{2k}) = p_{2k} = -i u_{2k}$. Again p_k is far from the neck between $\overline{\mathbb{C}}_{k-2}$ and $\overline{\mathbb{C}}_{k-1}$, and p_{2k} is far from the neck between $\overline{\mathbb{C}}_1$ and $\overline{\mathbb{C}}_2$, see Figure 4.

Hence we have defined an element $g \in \mathcal{W}(k)$ from the $4k-4$ complex parameters a_j, b_j, u_h. Since the roles of a_j and b_j are symmetric, the right parameters to consider if we want to parametrize the space of $g \in \mathcal{W}(k)$ around the Scherk limit are the u_h together

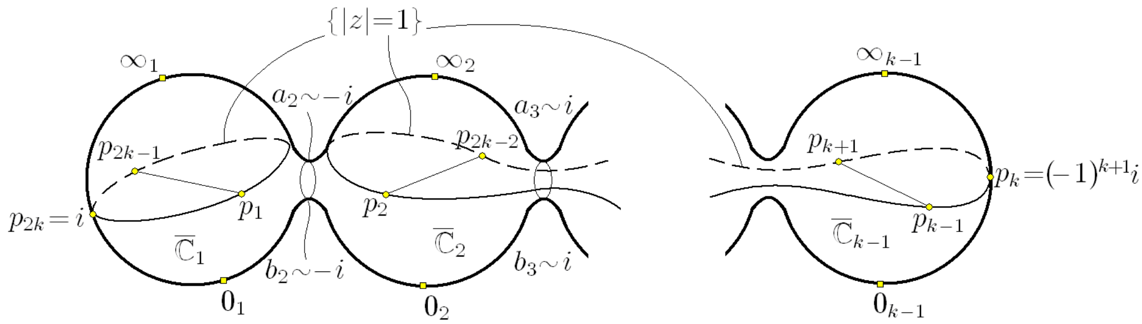

Figure 4: Parametrizing the Gauss map g for the particular case $u \in \mathcal{U}(k)$ close to $\mathbf{u}_\infty$. The points $p_j, p_{2k-j} \in \{|z| = 1\}$ are respectively close to $\alpha, -\alpha$ in each copy of $\overline{\mathbb{C}}$.

with the elementary symmetric functions of a_j, b_j. Let

$$x_j = \frac{a_j + b_j}{2}, \qquad y_j = a_j b_j, \qquad 1 < j < k.$$

We also exchange the variable y_j by $\zeta_j = y_j - x_j^2$ for $1 < j < k$ (the map $(x_j, y_j) \mapsto (x_j, \zeta_j)$ is a diffeomorphism). Next we introduce some notation. Given $\varepsilon > 0$, let

$$\begin{aligned}
&\mathbf{x}_\infty = (-i, i, \ldots, (-1)^k i),\ \mathbf{0} = (0 \ldots, 0) \in \mathbb{C}^{k-2},\\
&D(\mathbf{x}_\infty, \varepsilon) = \{\mathbf{x} \in \mathbb{C}^{k-2} \mid \|\mathbf{x} - \mathbf{x}_\infty\| < \varepsilon\},\\
&D(\mathbf{0}, \varepsilon) = \{\zeta \in \mathbb{C}^{k-2} \mid \|\zeta\| < \varepsilon\},\\
&D(\mathbf{u}_\infty, \varepsilon) = \{\mathbf{u} \in \mathbb{C}^{2k} \mid \|\mathbf{u} - \mathbf{u}_\infty\| < \varepsilon\},\\
&\mathcal{A} = \{\zeta \in D(\mathbf{0}, \varepsilon) \mid \zeta_j = 0 \ \text{ for some } j\}.
\end{aligned}$$

Since $\zeta_j = -\frac{1}{4}(a_j - b_j)^2$, the equation $\zeta_j = 0$ means that $a_j = b_j$ is a node, so $\mathcal{A}$ expresses when Σ pinches into a Riemann surface with nodes. It is easy to show that the map $\Theta : D(\mathbf{x}_\infty, \varepsilon) \times (D(\mathbf{0}, \varepsilon) \setminus \mathcal{A}) \times D(\mathbf{u}_\infty, \varepsilon) \to \mathcal{W}(k)$ defined by $\Theta(\mathbf{x}, \zeta, \mathbf{u}) = g$ is a local chart for $\mathcal{W}(k)$. Given $(\mathbf{x}, \zeta, \mathbf{u}) \in D(\mathbf{x}_\infty, \varepsilon) \times (D(\mathbf{0}, \varepsilon) \setminus \mathcal{A}) \times D(\mathbf{u}_\infty, \varepsilon)$, we call ϕ to the height differential associated to $g = \Theta(\mathbf{x}, \zeta, \mathbf{u})$ (see Section 3).

7.2 The equations.

Next we study the regularity of the induced metric by the pair (g, ϕ). Let $0_j, \infty_j$ be the points of Σ given by $z = 0$ and $z = \infty$ in $\overline{\mathbb{C}}_j$. The equations we have to solve are $\phi = 0$

at 0_j and ∞_j for $1 \le j < k$ (if which case ϕ clearly has necessarily simple zeros at $0_j, \infty_j$ because ϕ has $2k$ poles in Σ). These equations can be equivalently written as

$$\operatorname{Res}_{0_j}(g^{-1}\phi) = 0, \qquad \operatorname{Res}_{\infty_j}(g\phi) = 0, \qquad 1 \le j < k.$$

As was explained in Subsection 3.1, we may forget two of these equations by the residue theorem, so we will only solve these equations for $1 < j < k$.

We saw in Section 3 that the period problem at the ends writes as $|g(p_j)| = 1$ or equivalently $|\operatorname{Res}_{p_j}(g\phi)| = 1$ for $1 \le j \le 2k$ (these equalities are automatically satisfied when the components of $\mathbf{u}$ lie in $\mathbb{S}^1$ rather than only in $\mathbb{C}$).

Our next goal is to study the behavior of both g, ϕ and the equations we have to solve when $\zeta \in \mathcal{A}$. In this case, Σ decomposes into a number of spheres between 2 and $k-1$, g produces nonconstant meromorphic maps on them whose degrees add up to $k-1$ and the values of these meromorphic maps at the node points are $\pm i$.

7.3 Holomorphic extension of ϕ.

Given $(\mathbf{x}, \zeta, \mathbf{u}) \in D(\mathbf{x}_\infty, \varepsilon) \times (D(\mathbf{0}, \varepsilon) \setminus \mathcal{A}) \times D(\mathbf{u}_\infty, \varepsilon)$, let γ_j be a small circle enclosing the points a_j, b_j in $\overline{\mathbb{C}}_j$, with the positive orientation. By the Residue Theorem, we have

$$\frac{1}{2\pi i}\int_{\gamma_j} \phi = \operatorname{Res}_{p_{2k}}\phi + \sum_{\ell=1}^{j-1}(\operatorname{Res}_{p_\ell}\phi + \operatorname{Res}_{p_{2k-\ell}}\phi) = (-1)^{j+1} i. \tag{9}$$

When $a_j = b_j$, the Riemann surface Σ has a node at a_j. In this case the definition of ϕ must be changed as follows. For each double point $a_j = b_j$, we ask that ϕ has a simple pole at the point $a_j \in \overline{\mathbb{C}}_j$ with residue $(-1)^{j+1} i$ (this comes from equation (9)), and it has a simple pole at the point $a_j \in \overline{\mathbb{C}}_{j-1}$ with opposite residue.

Proposition 6 *ϕ depends holomorphically on all parameters $(\mathbf{x}, \zeta, \mathbf{u}) \in D(\mathbf{x}_\infty, \varepsilon) \times D(\mathbf{0}, \varepsilon) \times D(\mathbf{u}_\infty, \varepsilon)$ (including those tuples with $\zeta \in \mathcal{A}$). By this we mean that if $z \in \overline{\mathbb{C}}_j$ is away from the cuts and poles, then $\phi(z)/dz$ depends holomorphically on $(\mathbf{x}, \zeta, \mathbf{u})$.*

Proof. This result is standard; see e.g. Section 3.4 of [18] for Riemann surfaces with nodes and arbitrary genus. $\square$

Corollary 3 *For $1 \le j < k$, the maps $(\mathbf{x}, \zeta, \mathbf{u}) \in D(\mathbf{x}_\infty, \varepsilon) \times D(\mathbf{0}, \varepsilon) \times D(\mathbf{u}_\infty, \varepsilon) \mapsto \operatorname{Res}_{0_j}(g^{-1}\phi)$, $\operatorname{Res}_{\infty_j}(g\phi)$, $\operatorname{Res}_{p_j}(g\phi)$ are holomorphic.*

7.4 The modified ligature map.

Let $\widetilde{L} : D(\mathbf{x}_\infty, \varepsilon) \times D(\mathbf{0}, \varepsilon) \times D(\mathbf{u}_\infty, \varepsilon) \to \mathbb{C}^{4k-4}$ be the map defined by

$$\widetilde{L} = \left(\underbrace{\operatorname{Res}_{0_j}(g^{-1}\phi)}_{1<j<k}, \underbrace{\operatorname{Res}_{\infty_j}(g\phi)}_{1<j<k}, \underbrace{\operatorname{Res}_{p_j}(g\phi)}_{1\le j\le 2k} \right).$$

Corollary 3 implies that $\widetilde{L}$ is holomorphic. A straightforward computation gives that $\widetilde{L}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = (\mathbf{0}, \mathbf{0}, \mathbf{u}_\infty) \in (\mathbb{C}^{k-2})^2 \times \mathbb{C}^{2k}$.

Lemma 12 *With the notation above, the Jacobian $(2k-4)$-matrix*

$$\left. \frac{\partial \left(\operatorname{Res}_{0_j}(g^{-1}\phi), \operatorname{Res}_{\infty_j}(g\phi) \right)}{\partial(\mathbf{x}, \zeta)} \right|_{(\mathbf{x},\zeta,\mathbf{u})=(\mathbf{x}_\infty,\mathbf{0},\mathbf{u}_\infty)}$$

is regular.

Proof. We start with the partial derivatives of $\operatorname{Res}_{0_h}(g^{-1}\phi)$, $\operatorname{Res}_{\infty_h}(g\phi)$ with respect to the x_j variables. Then we can fix all the ζ_h as zero for $1 < h < k$, which means that Σ consists of copies $\overline{\mathbb{C}}_1, \dots, \overline{\mathbb{C}}_{k-1}$ of $\overline{\mathbb{C}}$ joined by $k-2$ nodes so that the node between $\overline{\mathbb{C}}_{j-1}$ and $\overline{\mathbb{C}}_j$ is placed at $x_j = a_j = b_j \sim (-1)^{j+1} i$ (here x_j must be thought as a variable) and the remaining nodes are all placed as $\pm i$. For $h \neq j-1, j$ fixed, the corresponding height differential ϕ in $\overline{\mathbb{C}}_h$ does not depend on x_j, hence

$$\frac{\partial \operatorname{Res}_{0_h}(g^{-1}\phi)}{\partial x_j}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = \frac{\partial \operatorname{Res}_{0_h}(g^{-1}\phi)}{\partial x_j}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = 0.$$

Concerning $\left.\frac{\partial}{\partial x_j}\right|_{(\mathbf{x}_\infty,\mathbf{0},\mathbf{u}_\infty)} \operatorname{Res}_{0_j}(g^{-1}\phi)$, the holomorphic extension of ϕ let us write

$$\phi = (-1)^j i \left(\frac{1}{z-p_j} + \frac{1}{z-p_{2k-j}} - \frac{1}{z-x_j} - \frac{1}{z-x_{\infty,j+1}} \right) \qquad \text{on } \overline{\mathbb{C}}_j,$$

where $x_{\infty,j+1} = (-1)^j i$ (when $j = k-1$, we should replace $x_{\infty,j+1}$ by $p_k = (-1)^{k+1} i$ in the above formula). This gives

$$\operatorname{Res}_{0_j}(g^{-1}\phi) = (-1)^j i \left(\frac{1}{x_j} + \frac{1}{x_{\infty,j+1}} \right), \qquad \operatorname{Res}_{\infty_j}(g\phi) = (-1)^j i (x_j + x_{\infty,j+1}),$$

hence

$$\frac{\partial \operatorname{Res}_{0_j}(g^{-1}\phi)}{\partial x_j}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = \frac{\partial \operatorname{Res}_{\infty_j}(g\phi)}{\partial x_j}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = (-1)^j i.$$

Analogously,

$$\frac{\partial \operatorname{Res}_{0_j}(g^{-1}\phi)}{\partial x_{j+1}}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = \frac{\partial \operatorname{Res}_{\infty_j}(g\phi)}{\partial x_{j+1}}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = (-1)^j i.$$

We now compute partial derivatives with respect to ζ_j, so we fix all the x_h as the corresponding component $x_{\infty,h}$ of $\mathbf{x}_\infty$, $\zeta_h = 0$ for all $h \neq j$ and think of ζ_j as a variable close to zero. This means that all Σ decomposes in $k-2$ spheres joined by nodes, $k-3$ of which correspond to single copies of $\overline{\mathbb{C}}_h$ (that we will call *simple spheres*) $h \neq j-1, j$, and just one sphere correspond to the copies $\overline{\mathbb{C}}_{j-1}, \overline{\mathbb{C}}_j$ (called a *double sphere*). On any single sphere, neither g nor ϕ depend on ζ_j hence

$$\frac{\partial \operatorname{Res}_{0_h}(g^{-1}\phi)}{\partial \zeta_j}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = \frac{\partial \operatorname{Res}_{\infty_h}(g\phi)}{\partial \zeta_j}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = 0, \quad h \neq j-1, j.$$

The remaining double sphere can be parametrized as

$$S = \{(z,w) \in (\overline{\mathbb{C}})^2 \mid w^2 = (z-a_j)(z-b_j) = (z - x_{\infty,j})^2 + \zeta_j\}.$$

The square root $w = \sqrt{(z-x_{\infty,j})^2 + \zeta_j}$ is well-defined on S, and we fix the sign of this square root as follows. Away from the cut we have $w^2 \sim (z - x_{\infty,j})^2$. We ask that $w \sim z - x_{\infty,j}$ in $\overline{\mathbb{C}}_j$, and consequently $z \sim -(z - x_{\infty,j})$ in $\overline{\mathbb{C}}_{j-1}$. Clearly g writes $g(z,w) = z$ on S, and an elementary argument using poles and residues leads to

$$\phi = 2(-1)^j i \left(\frac{w(p_j)}{z - z(p_j)} + \frac{w(p_{2k-j})}{z - z(p_{2k-j})} - \frac{w(x_{\infty,j+1})}{z - x_{\infty,j+1}} \right) \frac{dz}{w}.$$

This gives

$$\operatorname{Res}_{0_j}(g^{-1}\phi) = \frac{2(-1)^j i}{w(0)} \left(\frac{w(p_j)}{-z(p_j)} + \frac{w(p_{2k-j})}{-z(p_{2k-j})} + \frac{w(x_{\infty,j+1})}{x_{\infty,j+1}} \right),$$

$$\operatorname{Res}_{\infty_j}(g\phi) = 2(-1)^j i \Big(- w(p_j) - w(p_{2k-j}) + w(x_{\infty,j+1}) \Big).$$

The same formulae with opposite signs hold for the residues at 0_{j-1} and ∞_{j-1}, because of the valuation of the square root. Using that $x_{\infty,j+1} = (-1)^j i$ and $z(p_j) = (-1)^{j+1} i\alpha$ one easily computes

$$\frac{\partial \operatorname{Res}_{0_j}(g^{-1}\phi)}{\partial \zeta_j}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = \frac{5-\alpha^2}{2(\alpha^2-1)}, \qquad \frac{\partial \operatorname{Res}_{\infty_j}(g\phi)}{\partial \zeta_j}(\mathbf{x}_\infty, \mathbf{0}, \mathbf{u}_\infty) = \frac{3+\alpha^2}{2(\alpha^2-1)},$$

and the corresponding partial derivatives of the residues at 0_{j-1} and ∞_{j-1} only differ from the above ones in a sign. Now the regularity of the Jacobian matrix in Lemma 12 follows directly. $\square$

Proposition 7 *Let $\{M_n\}_n \subset \widetilde{\mathcal{S}}(k)$ be a sequence converging weakly to a Scherk limit. Then $M_n \in \widetilde{\mathcal{K}}(k)$ for n large enough.*

Proof. Without loss of generality we can assume that the fluxes $\mathbf{u}_n = F(M_n)$ converge to $\mathbf{u}_\infty = (\alpha, \overset{(k-1)}{\ldots}, \alpha, 1, -\alpha, \overset{(k-1)}{\ldots}, -\alpha, -1)$ with $\alpha \in \mathbb{S}^1 - \{\pm 1\}$. Since $\{M_n\}_n$ converges weakly to the Scherk limit associated to this angle α, we deduce that for n large enough, the element $g_n \in \mathcal{W}(k)$ that corresponds to M_n can be represented by $g_n = \Theta(\mathbf{x}_n, \zeta_n, \mathbf{u}_n)$ for certain $(\mathbf{x}_n, \zeta_n) \in D(\mathbf{x}_\infty, \varepsilon) \times (D(\mathbf{0}, \varepsilon) - \mathcal{A})$ (with the same notation of this Section). Furthermore $(\mathbf{x}_n, \zeta_n) \to (\mathbf{x}_\infty, \mathbf{0})$ as $n \to \infty$.

By Lemma 12 and the Implicit Function Theorem, there exists $\varepsilon' > 0$ small such that for any $\mathbf{u} \in D(\mathbf{u}_\infty, \varepsilon')$ there exists a unique pair $(\mathbf{x}(\mathbf{u}), \zeta(\mathbf{u})) \in D(\mathbf{x}_\infty, \varepsilon') \times D(\mathbf{0}, \varepsilon')$ so that $\widetilde{L}(\mathbf{x}(\mathbf{u}), \zeta(\mathbf{u}), \mathbf{u}) = (0, 0, \mathbf{u})$. Note that we do not know if some components of $\zeta(\mathbf{u})$ vanish, so the Riemann surface associated to $(\mathbf{x}(\mathbf{u}), \zeta(\mathbf{u}), \mathbf{u})$ might have nodes. But this proves that for any $\mathbf{u} \in D(\mathbf{u}_\infty, \varepsilon')$, there is (locally) at most one marked minimal surface $M(\mathbf{u}) \in \widetilde{\mathcal{S}}(k)$ with $F(M(\mathbf{u})) = \mathbf{u}$. On the other hand, for $\mathbf{u} \in \mathcal{U}(k) \setminus \mathcal{U}_0(k)$ there exists exactly one Karcher saddle tower $M_{\mathbf{u}} \in \widetilde{\mathcal{K}}(k)$ with $F(M_{\mathbf{u}}) = \mathbf{u}$. Hence if $\mathbf{u}_n = F(M_n)$ lies in $\mathcal{U}(k) \setminus \mathcal{U}_0(k)$ for n large enough, then Proposition 7 clearly holds.

It only remains to analyze the case that after passing to a subsequence, $\mathbf{u}_n \in \mathcal{U}_0(k)$ for all n. In this case, we can write $\mathbf{u}_n = (\alpha_n, \overset{(k-1)}{\ldots}, \alpha_n, 1, -\alpha_n, \overset{(k-1)}{\ldots}, -\alpha_n, -1)$ with $\alpha_n \to \alpha$ as $n \to \infty$. If we take $\zeta_j = 0$ and $x_j(n) = (-1)^{j+1}\alpha_n$ for all j, then the previous computations in this section give that $\widetilde{L}(\mathbf{x}_n, \mathbf{0}, \mathbf{u}_n) = (\mathbf{0}, \mathbf{0}, \mathbf{u}_n)$ for all n, where $\mathbf{x}_n = (x_2(n), \ldots, x_{k-1}(n))$. Geometrically this tuple $(\mathbf{x}_n, \mathbf{0}, \mathbf{u}_n)$ also represents a Scherk limit, with $k-1$ Scherk surfaces whose fluxes at the ends are ± 1, $\pm\alpha_n$. In particular, $(\mathbf{x}_n, \mathbf{0}, \mathbf{u}_n)$ does not represent a marked surface in $\widetilde{\mathcal{S}}(k)$, hence locally there are not marked surfaces with flux equal to $\mathbf{u}_n$, a contradiction. $\square$

We finish this section by proving Proposition 3. Note that this proposition was not used to prove that $\widetilde{\mathcal{S}}_0(k)$ is empty, so we may assume that this has already been proven (see section 4). Choose an element $\mathbf{u}_* \in \mathcal{U}_0(k) \setminus \{\mathbf{u}_0(k)\}$. If Proposition 3 does not hold, then there exists a sequence $\{M_n\}_n \subset \widetilde{\mathcal{S}}(k) \setminus \widetilde{\mathcal{K}}(k)$ such that $\{u_n = F(M_n)\}_n \to \mathbf{u}_*$. By

Proposition 5 **(B)** and since $\widetilde{\mathcal{S}}_0(k)$ is empty, $\{M_n\}_n$ converges weakly to a Scherk limit. But this contradicts Proposition 7.

8 Uniqueness around the catenoid limit.

In this section we will study *special* surfaces close to the catenoid limit and conclude that a sequence of special surfaces cannot converge to the catenoid limit if $k \geq 3$. This was needed in the proof of Proposition 1.

Geometrically, the catenoid limit is the limit behavior of the Scherk limit when $\alpha \to 1$, with α as in Section 7. If we follow the line of arguments of that section, what goes wrong is that when $\alpha \to 1$, the poles of ϕ converge to the nodes, and this is bad to control the limit of ϕ. For this reason, the geometric setup in this section if quite different from the previous sections. We consider only special surfaces. We rotate the surface so that all ends but two are horizontal. We call the remaining two ends the *top* and *bottom* ends. Now ϕ has only two poles at the top and bottom ends, and no poles at the horizontal ends. We scale the surface so that the vertical part of the flux on any horizontal section is, up to sign, equal to 2π. We are now in a situation very similar to Section 7 of [12] and we will follow the same arguments there up to some minor modifications.

8.1 Weierstrass data.

We now write a model for the Weierstrass data of a special surface close to the catenoidal limit in the above geometrical setup. Consider $k-1$ copies $\overline{\mathbb{C}}_1, \ldots, \overline{\mathbb{C}}_{k-1}$ of the Riemann sphere $\overline{\mathbb{C}}$, and distinct complex numbers a_j, b_j, $1 < j < k$, in a punctured neighborhood of 0 if j is odd and of ∞ if j is even. Glue $\overline{\mathbb{C}}_{j-1}$ with $\overline{\mathbb{C}}_j$ along the cut from a_j to b_j in the usual way. This produces a Riemann surface Σ of genus 0 together with a meromorphic function $g : \Sigma \to \overline{\mathbb{C}}$ defined by $g = z$ in each $\overline{\mathbb{C}}_j$. Its branch values are a_j, b_j for $1 < j < k$.

The horizontal ends are at the $2(k-1)$ points of Σ where $g = 0$ or $g = \infty$. We call 0_j and ∞_j the points 0 and ∞ in $\overline{\mathbb{C}}_j$. The bottom and top ends are respectively some points in $\overline{\mathbb{C}}_1$ and $\overline{\mathbb{C}}_{k-1}$. We may orient the surface so that the Gauss map at the bottom end is some nonzero complex number α close to 0 (so the bottom end is the point $z = \alpha$ in $\overline{\mathbb{C}}_1$). If k is odd, the Gauss map at the top end is some nonzero complex number β close to 0. If k is even, the Gauss map at the top end is of the form $1/\beta$ with $\beta \in \mathbb{C} - \{0\}$, β close to 0 (see Figure 5). *A priori,* we do not impose any relation between α and β. Of course,

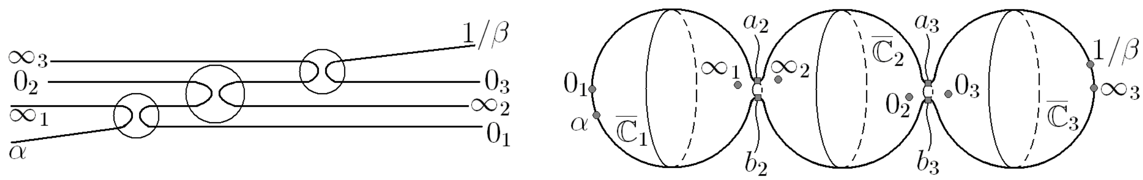

Figure 5: Left: A special surface M close to the catenoid limit in the case $k = 4$, with the value of the Gauss map at the ends. Right: The Riemann surface corresponding to M, with the ends and the branch points of the Gauss map.

both complex numbers are related by flux arguments. We will expose this connection in Subsection 8.2.

The height differential ϕ is defined as the unique meromorphic 1-form on Σ with simple poles at the bottom and top ends, and respective residues 1 and -1. Let Γ_j be the unit circle in $\overline{\mathbb{C}}_j$ with the positive orientation if j is odd and the negative orientation if j is even. All these curves represent the same homology class in Σ and $\int_{\Gamma_j} \phi = 2\pi i$.

Note that the roles of a_j and b_j, $1 < j < k$ are symmetric, so the right parameters are their elementary symmetric functions: we introduce the parameters

$$x_j = \frac{1}{2}(a_j + b_j), \qquad y_j = a_j b_j \qquad \text{if } j \text{ is odd}$$

$$x_j = \frac{1}{2}\left(\frac{1}{a_j} + \frac{1}{b_j}\right), \qquad y_j = \frac{1}{a_j b_j} \qquad \text{if } j \text{ is even}$$

so all parameters α, β, x_j, y_j, $1 < j < k$, are close to 0. As usual we write $\mathbf{x} = (x_2, \ldots, x_{k-1})$ and $\mathbf{y} = (y_2, \ldots, y_{k-1})$.

8.2 The equations.

The period problem reduces to impose that the period at each end is the same up to sign. This means that there exists $\lambda \in \mathbb{C}$ such that at each end,

$$\overline{\text{Res } (g^{-1}\phi)} + \text{Res } (g\phi) = \pm\lambda.$$

The period is then $T = \pm\pi(\mathrm{Im}\lambda, -\mathrm{Re}\lambda, 0)$. We also know that the signs $\pm$ alternate at consecutive ends. This gives the following system of equations:

$$\begin{cases} \dfrac{1}{\overline{\alpha}} + \alpha = \lambda \\ \dfrac{1}{\overline{\beta}} + \beta = \begin{cases} \lambda & (k \text{ odd}) \\ -\overline{\lambda} & (k \text{ even}) \end{cases} \\ \mathrm{Res}_{\,0_j}\left(g^{-1}\phi\right) = (-1)^j \overline{\lambda} \qquad 1 \leq j \leq k-1 \\ \mathrm{Res}_{\,\infty_j}\left(g\phi\right) = (-1)^j \lambda \qquad 1 \leq j \leq k-1. \end{cases} \tag{10}$$

Using the Residue Theorem, these equations imply that

$$\int_{\Gamma_j} g\phi = \int_{\Gamma_1} g\phi = 2\pi i \,\mathrm{Res}_{\,\alpha}\,(g\phi) = 2\pi i \alpha,$$

$$\int_{\Gamma_j} g^{-1}\phi = \int_{\Gamma_1} g^{-1}\phi = 2\pi i \left(\mathrm{Res}_{\,\alpha}\left(g^{-1}\phi\right) + \mathrm{Res}_{\,0_1}\left(g^{-1}\phi\right)\right) = 2\pi i \left(\frac{1}{\alpha} - \overline{\lambda}\right) = -2\pi i \overline{\alpha}.$$

Note that $\int_{\Gamma_j} g^{-1}\phi$ is conjugate to $\int_{\Gamma_j} g\phi$ which means that Γ_j is a closed curve on the surface, as expected. For $1 \leq j < k$ let

$$A_j = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_j} g^{-1}\phi & (j \text{ odd}), \\ \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_j} g\phi & (j \text{ even}). \end{cases}$$

For $1 < j < k$ let

$$B_j = \begin{cases} \mathrm{Res}_{\,0_{j-1}}\left(g^{-1}\phi\right) \cdot \mathrm{Res}_{\,0_j}\left(g^{-1}\phi\right) & (j \text{ odd}), \\ \mathrm{Res}_{\,\infty_{j-1}}\left(g\phi\right) \cdot \mathrm{Res}_{\,\infty_j}\left(g\phi\right) & (j \text{ even}). \end{cases}$$

Then equation (10) implies

$$\begin{cases} \beta = \begin{cases} \alpha & (k \text{ odd}) \\ -\overline{\alpha} & (k \text{ even}) \end{cases} \\ \text{For all } \ 1 \leq j \leq k-2, \quad A_j = \begin{cases} -\overline{\alpha} & (j \text{ odd}) \\ \alpha & (j \text{ even}) \end{cases} \\ \text{For all } \ 1 < j < k, \quad B_j = \begin{cases} -\overline{\lambda}^2 & (j \text{ odd}) \\ -\lambda^2 & (j \text{ even}) \end{cases} \qquad \text{with } \lambda = \dfrac{1}{\overline{\alpha}} + \alpha. \end{cases} \tag{11}$$

In fact, the system of equations (10) is equivalent to (11), but we will not need that since we want to prove a non-existence result. Note that (11) is a system of $2(k-1)$ complex equations in the $2(k-1)$ variables α, β, a_j, b_j, $1 < j < k$. The goal is to prove that for $\alpha \neq 0$ close to 0, the system (11) has no solutions representing a special surface.

8.3 Holomorphic extension of ϕ.

When $a_j = b_j$ for some j, the definition of g gives a Riemann surface with a node between $\mathbb{C}_{j-1}$ and $\mathbb{C}_j$. In this case, ϕ needs two more simple poles at $z = a_j$ in $\mathbb{C}_j$ and in $\mathbb{C}_{j-1}$. The residues at these poles are determined by $\int_{\Gamma_j} \phi = 2\pi i$. Then as in Section 7 we have

Proposition 8 *ϕ depends holomorphically on $(\alpha, \beta, \mathbf{x}, \mathbf{y})$ in a neighborhood of $(0, 0, \mathbf{0}, \mathbf{0})$.*

Proof. Same as Proposition 6. □

Proposition 9 *For $1 \leq j \leq k-2$, the function A_j extends as a holomorphic function of $(\alpha, \beta, \mathbf{x}, \mathbf{y})$ in a neighborhood of $(0, 0, \mathbf{0}, \mathbf{0})$. The same holds with the function $\widetilde{B}_j = y_j B_j$ for $1 < j < k$.*

Proof. The first point is a consequence of the previous proposition, since the curve Γ_j stays in the limit Riemann surface minus its nodes. The second point does not follow from a similar argument because the points $0_{j-1}, 0_j$ (resp. ∞_{j-1}, ∞_j) collapse into node points. Instead, we need to control the rate at which the residues in the definition of B_j blow-up. The reader can find this estimate in the proof of Proposition 10 in [12]. □

Remark 6 $\mathrm{Res}_{0_j}(g^{-1}\phi)$ *is a multi-valued function of the parameters. The reason for this is that the points 0_j and 0_{j-1} are close to the branch points a_j, b_j. When the parameters a_j and b_j vary, the cut from a_j to b_j may cross 0, in which case 0_j and 0_{j-1} exchange sheets so do not depend continuously on the parameters as points on the Riemann surface. On the other hand, the unordered pair $\{0_{j-1}, 0_j\}$ depends continuously on the parameters, which is why the symmetric functions of the residues at 0_{j-1} and 0_j are well-defined functions. This is the main reason why we introduced the functions B_j. See also remark 9 in [12].*

8.4 Partial derivatives.

Proposition 10 *For each $j = 2, \ldots, k-1$, it holds*

$$A_{j-1}(0,0,\mathbf{0},\mathbf{0}) = 0, \qquad \widetilde{B}_j(0,0,\mathbf{0},\mathbf{0}) = -1, \qquad \frac{\partial A_{j-1}}{\partial x_j}(0,0,\mathbf{0},\mathbf{0}) = 1.$$

All remaining partial derivatives of the A_i with respect to x_j, y_j are zero. We will not need the partial derivatives of the $\widetilde{B}_i$.

Proof. By Proposition 9, for j fixed we can compute either the value of $A_{j-1}, \widetilde{B}_j$ or that of their partial derivatives with respect to x_j, y_j by assuming $\alpha = \beta = 0$ and $x_h = y_h = 0$ for all $h \neq j$. The associated Riemann surface has $k-3$ nodes which disconnect it into $k-2$ genus zero components. In each component, the height differential has two simple poles with residue ± 1. These poles are either nodes or the top end or the bottom end.

On $k-3$ of these components (which we call *simple spheres*) the height differential ϕ is dz/z. The remaining component S corresponds to $\overline{\mathbb{C}}_{j-1}$ glued with $\overline{\mathbb{C}}_j$ (we call it a *double sphere*).

First consider the case j odd. Then S can be parametrized by $\{(z,w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z-a)(z-b)\}$ where $a+b = 2x_j$, $ab = y_j$, so that $w = \sqrt{(z-a)(z-b)}$ is well-defined on S. We fix the sign of the square root by asking that $w \sim z$ in $\overline{\mathbb{C}}_j$ and $w \sim -z$ in $\overline{\mathbb{C}}_{j-1}$. Now ϕ has simple poles at ∞_{j-1} and ∞_j with respective residues 1 and -1. As

$$\operatorname{Res}_{\infty_j} \frac{dz}{w} = \operatorname{Res}_{\infty} \frac{dz}{z\sqrt{1 - 2x_j/z + y_j/z^2}} = -1,$$

we conclude that $\phi = dz/w$. Thus

$$A_j = A_j(x_j, y_j) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{dz}{zw} = -\operatorname{Res}_{\infty_j} \frac{dz}{zw} = 0,$$

$$A_{j-1}(x_j, y_j) = \frac{1}{2\pi i} \int_{\Gamma_{j-1}} \frac{z dz}{w} = \operatorname{Res}_{\infty_{j-1}} \frac{z\, dz}{-z\sqrt{1 - 2x_j/z + y_j/z^2}} = x_j.$$

Concerning $\widetilde{B}_j$, we write

$$\operatorname{Res}_{0_j} \left(g^{-1}\phi\right) = \frac{1}{\sqrt{y_j}}, \qquad \operatorname{Res}_{0_{j-1}} (g\phi) = \frac{-1}{\sqrt{y_j}}.$$

The computations in the case j even are similar with the following modifications

$$w = \sqrt{\left(\frac{1}{z} - \frac{1}{a}\right)\left(\frac{1}{z} - \frac{1}{b}\right)}, \qquad \phi = \frac{-dz}{z^2 w},$$

$$A_j(x_j, y_j) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{-dz}{zw} = \operatorname{Res}_{0_j} \frac{dz}{zw} = 0,$$

$$A_{j-1}(x_j, y_j) = \frac{1}{2\pi i} \int_{\Gamma_{j-1}} \frac{-dz}{z^3 w} = \operatorname{Res}_{0_{j-1}} \frac{-dz}{-z^2\sqrt{1 - 2x_j z + y_j z^2}} = x_j,$$

which finishes the proof of the proposition. $\square$

8.5 Inverse Function Theorem.

Note that $1/B_j = y_j/\widetilde{B}_j$. Since $\widetilde{B}_j(0,0,\mathbf{0},\mathbf{0}) = -1$, the function $1/B_j$ extends holomorphically to a neighborhood of $(0,0,\mathbf{0},\mathbf{0})$. Moreover

$$\frac{\partial B_j^{-1}}{\partial x_i}(0,0,\mathbf{0},\mathbf{0}) = 0, \qquad \frac{\partial B_j^{-1}}{\partial y_i}(0,0,\mathbf{0},\mathbf{0}) = -\delta_{ij}.$$

Define

$$\Theta(\alpha,\beta,\mathbf{x},\mathbf{y}) = \left(\alpha,\beta,A_1,\ldots,A_{k-2},\frac{1}{B_2},\ldots,\frac{1}{B_{k-1}}\right).$$

Then Θ is holomorphic in a neighborhood of $(0,0,\mathbf{0},\mathbf{0})$ and its Jacobian matrix at $(0,0,\mathbf{0},\mathbf{0})$ is invertible. By the Inverse Function Theorem, Θ is a biholomorphism from a neighborhood of $(0,0,\mathbf{0},\mathbf{0})$ onto its image. Hence for $t \in \mathbb{C}$ close to 0, there exists a unique $(\alpha,\beta,\mathbf{x},\mathbf{y})$ close to $(0,0,\mathbf{0},\mathbf{0})$ such that $\alpha = t$ and β, A_j, B_j have the values given by the system (11).

We now remark that (11) has an obvious solution: For $t \in \mathbb{C} - \{0\}$ take

$$\alpha = t, \qquad \beta = \begin{cases} t & (k \text{ odd}) \\ -\overline{t} & (k \text{ even}) \end{cases} \qquad a_j = b_j = \begin{cases} t & (j \text{ odd}) \\ -\frac{1}{\overline{t}} & (j \text{ even}) \end{cases}$$

The corresponding Riemann surface has $k-2$ nodes which disconnect it into $k-1$ simple spheres. On each sphere $\overline{\mathbb{C}}_j$ we have

$$g = z, \qquad \phi = \frac{dz}{z-t} - \frac{dz}{z+\frac{1}{\overline{t}}}.$$

This is the Weierstrass representation of a singly periodic Scherk minimal surface. It is straightforward to check (and geometrically clear) that equation (11) is satisfied. Geometrically, this solution is a Scherk limit, namely a configuration of $k-1$ Scherk surfaces. It is not a true minimal surface (unless, of course, $k=2$). By uniqueness we conclude that (11) has no other solutions $(\alpha,\beta,\mathbf{x},\mathbf{y})$ around $(0,0,\mathbf{0},\mathbf{0})$, which means that there are no special surfaces close to the catenoid limit if $k \geq 3$.

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