HOLLOW VORTICES AND MINIMAL SURFACES

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Abstract: We consider an overdetermined elliptic problem known as the hollow vortex problem. We prove that the solutions to this problem are in 1:1 correspondence with minimal graphs bounded by horizontal symmetry lines. We use this correspondence to give various examples of domains with hollow vortices.

MSC-classification: primary 35N25, secondary 53A10.

1. Introduction

We consider the following overdetermined problem in the plane, known as the *hollow vortex problem*:

$$
(1) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u \text{ constant} & \text{on each component of } \partial\Omega \\ ||\nabla u|| = 1 & \text{on } \partial\Omega \end{cases}
$$

Here $\Omega \subset \mathbb{R}^2$ is an unbounded domain with smooth, non-empty boundary, and $u : \Omega \to \mathbb{R}$ is a smooth function. Observe that we require u to be constant on each boundary component, but we do not ask that the constant is the same for all boundary components. The fact that $||\nabla u|| = 1$ on the boundary is equivalent to the Neumann condition $\frac{\partial u}{\partial \nu} = \pm 1$, where ν is the (interior) unit normal to the boundary. Again, the sign of $\frac{\partial u}{\partial \nu}$ may depend on the boundary component.

Problem (1) is overdetermined because both Neumann and Dirichlet boundary conditions are prescribed, which is not possible for general domains Ω. A domain Ω admitting a function u solving Problem (1) will be called a *domain with hollow vortices*. The hollow vortices refer to the components of $\mathbb{R}^2 \setminus \Omega$. The name comes from the following physical interpretation of Problem (1): the stationary flow of an inviscid, incompressible fluid in a domain Ω is described by Euler equations:

$$
\text{div } \vec{v} = 0, \qquad (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p
$$

FIGURE 1. A Von Karman vortex street, from Van Dyke Album of Fluid Motion [16]

where $\vec{v}$ denotes the velocity vector, p the pressure and ρ the mass density of the fluid. In the 2-dimensional case, we can write $\vec{v} = \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right)$ for some function u called the stream function. If we assume moreover that the flow is irrotational, then $\Delta u = 0$. The condition that u is constant on a boundary component γ of Ω means that γ is a stream line. The condition that $||\nabla u||$ is constant on γ means that the norm of the velocity is constant, which is equivalent to constant pressure by Bernoulli law. It is crucial for the work in this paper that the constant is the same for all boundary components. That constant is chosen to be equal to 1 by scaling.

We can think of γ as bounding a spinning bubble of air with constant pressure inside, or "hollow vortex". Hollow vortices have been proposed as a model for some periodic configurations of vortices observed in the turbulent flow past an obstacle known as Von Karman vortex streets. The authors of [3] have found hollow vortex solutions corresponding precisely to what is beeing observed in Figure 1.

In the particular case where $u = 0$ on $\partial\Omega$ and $u > 0$ in Ω, solutions to the hollow vortex problem have been studied in [6], [10] and completely classified by the author in [14], by establishing a correspondence with a certain type of minimal surfaces and using classification results in minimal surface theory. It turns out that the correspondence extends to the general case of Problem (1), only to a wider class of minimal surfaces. Our goal in this paper is to describe this correspondence and give examples.

The corresponding overdetermined problem for minimal surfaces is the following:

$$(2) \quad \begin{cases} (1 + v_y^2)v_{xx} + (1 + v_x^2)v_{yy} - 2v_x v_y v_{xy} = 0 & \text{in } \widehat{\Omega} \\ v \text{ constant} & \text{on each component of } \partial\widehat{\Omega} \\ ||\nabla v|| \to \infty & \text{on } \partial\widehat{\Omega} \end{cases}$$

Here $\widehat{\Omega} \subset \mathbb{R}^2$ is an unbounded domain with non-empty boundary and $v : \widehat{\Omega} \to \mathbb{R}$ is a smooth function. The subscripts denote partial derivatives. The first equation is the minimal surface equation: it says that the graph of v, denoted M, is a minimal surface. The limit in the last condition means the following: for any $z_0 \in \partial\widehat{\Omega}$, $\lim_{z \to z_0} \frac{\partial v}{\partial \nu} = \pm\infty$, the

limit being uniform on compact sets of $\partial\widehat{\Omega}$. Geometrically speaking, this means that the Gauss map (i.e. the unit normal vector) of M is horizontal on the boundary. Such minimal surfaces can be smoothly extended by reflection in the horizontal plane containing each boundary component. For this reason, we call them *minimal graphs bounded by horizontal symmetry curves*.

For simplicity, we assume that all domains Ω and $\widehat{\Omega}$ considered in this paper satisfy the following finiteness hypothesis:

Hypothesis 1. *Either:*
- *Ω has a finite number of boundary components,*
- *or Ω is invariant by a translation T and the quotient Ω/T has a finite number of boundary components (simply-periodic case),*
- *or Ω is invariant by two independent translations (doubly-periodic case).*

Under this hypothesis, the main result of this paper is

Theorem 1. *There is a 1:1 correspondence between:*
- *solutions (Ω, u) of Problem (1) such that $||\nabla u|| < 1$ in Ω,*
- *solutions $(\widehat{\Omega}, v)$ of Problem (2).*

We describe how the correspondence works in Section 2. An interesting feature of the correspondence is that the domain with hollow vortices Ω and the corresponding minimal graph M are conformally related. Also, each component of $\partial\widehat{\Omega}$ is a translation of the corresponding component of $\partial\Omega$.

If Ω is a domain with hollow vortices, then in general, extending the corresponding minimal surface M by reflection will not give an embedded minimal surface. However, there are two particular cases where M can be extended to a complete embedded minimal surface:

- Case I: $u > 0$ in Ω and $u = 0$ on $\partial\Omega$ (this is the case already considered in [14]). In this case, the corresponding minimal surface M lies in the half-space $x_3 > 0$ and has boundary in the horizontal plane $x_3 = 0$. It can be extended by reflection to a complete, embedded minimal surface.
- Case II: $0 < u < c$ in Ω and $u = 0$ or $u = c$ on each boundary component of $\partial\Omega$. In this case, M lies in the slab $0 < x_3 < c$ and has boundary in the horizontal planes at height 0 and c. It can be extended by iterated reflections in horizontal planes into a complete, embedded, periodic minimal surface with vertical period $2c$.

We know a lot of such minimal surfaces, including some triply-periodic minimal surfaces discovered in the 19th century by Schwarz. We will review some of the classical examples in Section 3.

For some reason, specialists in the field of minimal surfaces are mostly interested in embedded surfaces, so the known examples only correspond to domains with hollow vortices of type I or II. However, some interesting methods have been developed to construct minimal surfaces, for example: the *conjugate Plateau construction*, see H. Karcher [9], or the *flat structure method* of M. Weber and M. Wolf [17]. Relaxing the constraint that we want the minimal surface to be embedded, it should be possible to adapt these methods to construct more general domains with hollow vortices.

In Section 4, we discuss another method, which was developed by the author and collaborators to construct minimal surfaces with small catenoidal necks. We will see how it can be adapted to construct domains with small hollow vortices.

1.1. **Related works.** A family of periodic solutions to the hollow vortex problem was constructed by Baker, Saffman and Sheffield in [1]. It corresponds to the family of horizontal Scherk surfaces (see Figure 2, top left) – the authors were of course not aware of that relationship. The same solution was derived again by Crowdy and Green in [3], together with another family of solutions called "staggered vortex streets". The corresponding minimal graphs are periodic and take on two different values on the boundary as in Case II. However, they are asymptotic to half-planes of non-zero slope at infinity, so extending these surfaces by reflection yields complete minimal surfaces which are not embedded.

Under the additional assumption that $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, solutions to the hollow vortex problem are called exceptional domains. Hauswirth, Hélein and Pacard studied the problem in [6] and discovered an exceptional domain which, as it turns out, corresponds to the horizontal catenoid. Partial classification results were obtained by Khavinson, Lundberg and Teodorescu in [10]. A complete classification is given in [14].

In a recent paper, Eremenko and Lundberg [5] have investigated the hollow vortex problem under the assumption that $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$. Solutions are called quasi-exceptional domains. Two examples are constructed. The first one corresponds to the minimal surface one gets if one tries to add a vertical handle to the horizontal catenoid. The second one corresponds to the minimal surface one gets if one tries to deform the horizontal Scherk surface so that the "holes" have different sizes. Both constructions are known to fail because one cannot solve the vertical period problem. On the hollow vortex side, this means that the function u takes on different values on the boundary components, which is perfectly fine.

In [4], the authors compute solutions to the hollow vortex problem which are a mixture of smooth hollow vortices and point-vortices. These point-vortices can probably be regularized into small hollow vortices in the spirit of what we do in Section 4, but the resulting solution will not solve Problem (1) because $||\nabla u||$ will take on different constant values on the different boundary components (it will be very large on the boundary of the small hollow vortices). This more general hollow vortex problem, where the constant value of $||\nabla u||$ depends on the boundary component, has been studied by many authors.

Solutions do not correspond to minimal surfaces, at least not in the way described in this paper.

2. THE CORRESPONDENCE

2.1. **Preliminary observations.** We first discuss the hypothesis $||\nabla u|| < 1$ in the statement of Theorem 1. Let (Ω, u) be a solution to the hollow vortex problem (1). The function $u_z = \frac{1}{2}(u_x - \mathrm{i}u_y)$ is holomorphic in Ω and satisfies $|2u_z| = 1$ on $\partial\Omega$. Let us rule out the trivial case where u_z is constant, in which case Ω is a half-plane or a band bounded by two parallel lines. It is known that $||\nabla u|| < 1$ in Ω in the following cases:

(1) if Ω and u_z are doubly periodic, by the maximum principle for holomorphic functions in the quotient,
(2) if $||\nabla u||$ is bounded, by a Phragmen Lindelöf-type result of Fuchs [8],
(3) if u is bounded from below (or above) in Ω, by the proof of Lemma 2 in [5]. (In this paper, the authors assume that $\frac{\partial u}{\partial \nu} = +1$ on the boundary, but only the condition $||\nabla u|| = 1$ is used in the proof of Lemma 2.)

Let us also mention that provided $||\nabla u|| < 1$, the domain Ω must be strictly concave: see the proof of Proposition 4 in [14].

2.2. **Weierstrass representation.** For the reader not familiar with minimal surfaces and to fix notations, we recall the Weierstrass representation formula:

$$(3) \qquad X(z) = (x_1(z), x_2(z), x_3(z)) = X_0 + \mathrm{Re} \int_{z_0}^{z} \left(\frac{1}{2}(g^{-1} - g)\omega, \frac{\mathrm{i}}{2}(g^{-1} + g)\omega, \omega \right)$$

In its local form, g is a meromorphic function on a simply connected domain $\Sigma \subset \mathbb{C}$ and $\omega = f(z)dz$ where f is a holomorphic function on Σ having a zero at each zero or pole of g, with the same multiplicity. $z_0 \in \Sigma$ is an arbitrary base point and X_0 is some constant vector. Then $X : \Sigma \to \mathbb{R}^3$ is a conformal parametrization of a minimal surface M. Moreover, the Gauss map of M is given by

$$N = \left(\frac{2\,\mathrm{Re}(g)}{|g|^2 + 1}, \frac{2\,\mathrm{Im}(g)}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1} \right).$$

In other words, g is the stereographic projection of the Gauss map.

In its global form, Σ is a Riemann surface, g is a meromorphic function and ω is a holomorphic 1-form on Σ.

2.3. **The correspondence "vortex $\to$ minimal".** Let (Ω, u) be a solution to the hollow vortex problem (1). Consider the minimal surface M given by the Weierstrass representation formula (3) with

$$g = \frac{-1}{2u_z}, \quad \omega = 2u_z\,dz, \quad X_0 = (0, 0, u(z_0)).$$

Let $\psi(z) = x_1(z) + \mathrm{i}x_2(z) : \Omega \to \mathbb{C}$.

Proposition 1. *In the above setup:*

(1) $x_3(z) = u(z)$.
(2) $\psi(z)$ *is well defined in* Ω, *namely does not depend on the integration path from* z_0 *to* z.
(3) $d\psi = dz$ *along* $\partial\Omega$.

Assume moreover that $||\nabla u|| < 1$ *in* Ω. *Then*

(4) *For any* $z' \neq z$ *in* Ω, $0 < |\psi(z') - \psi(z)| < |z' - z|$.
(5) ψ *is a diffeomorphism from* Ω *to* $\widehat{\Omega} = \psi(\Omega)$.
(6) *The boundary of* $\widehat{\Omega}$ *is* $\psi(\partial\Omega)$.

The proof of this proposition is essentially the same as the proof of Theorem 9 in [14]. For completeness, we give the details in Appendix A.

From Points (1) and (5), we see that M is the graph of $v = u \circ \psi^{-1}$ over the domain $\widehat{\Omega}$. Since $|g| = 1$ on $\partial\Omega$, the Gauss map is horizontal on the boundary, so $||\nabla v|| \to \infty$ on $\partial\widehat{\Omega}$. From Point (2), we see that each component of $\partial\widehat{\Omega}$ is a translation of the corresponding component of $\partial\Omega$. Moreover, from Point (4), we see that ψ moves the boundary components toward each other.

Remark 1. *In* [14] *we took* $g = 2u_z$, *so* $|g| < 1$ *in* Ω *and the Gauss map of* M *was pointing down. For a graph it is more natural to choose the upward pointing normal so that the horizontal projection preserves orientation. The correspondence has better properties with the antipodal choice* $g = \frac{-1}{2u_z}$.

2.4. **The correspondence "minimal $\to$ vortex".** Let $(\widehat{\Omega}, v)$ be a solution to Problem (2) and M be the minimal surface given as the graph of v. We orient M by its upward pointing normal. Then M is parametrized on some Riemann surface (with boundary) Σ by the Weierstrass representation formula (3). We have $|g| > 1$ on Σ and $|g| = 1$ on $\partial\Sigma$. We observe that even though Σ is diffeomorphic to the planar domain $\widehat{\Omega}$, in practice, it will not be given explicitly as a domain in the plane (see examples in Section 3), so it is better to leave it as an abstract Riemann surface. Let $\psi(z) = x_1(z) + ix_2(z)$. Since M is a graph, ψ is a diffeomorphism from Σ to $\widehat{\Omega}$. Define $F : \widehat{\Omega} \to \mathbb{C}$ by

$$F(\psi(z)) = -\int_{z_0}^{z} g\omega.$$

Proposition 2. *In the above setup:*

(1) F *is well defined in* $\widehat{\Omega}$.
(2) $dF = dz$ *on* $\partial\widehat{\Omega}$.
(3) *For any* $z \neq z'$ *in* $\widehat{\Omega}$, $|F(z) - F(z')| > |z - z'|$.
(4) F *is a diffeomorphism from* $\widehat{\Omega}$ *to* $\Omega = F(\widehat{\Omega})$.
(5) *The function* $u(z) = v(F^{-1}(z))$ *solves Problem* (1) *and satisfies* $||\nabla u|| < 1$ *in* Ω.

The proof of this proposition is essentially the same as the proof of Theorem 10 in [14]. For completeness, we give the details in Appendix B.

The maps $(\Omega, u) \mapsto (\widehat{\Omega}, v)$ and $(\widehat{\Omega}, v) \mapsto (\Omega, u)$ defined by Propositions 1 and 2 are inverse of each other, provided we identify two domains which differ by a translation. See Theorem 11 in [14].

Remark 2. *A computation shows that dF is given in term of v by*

$$dF = dx + \mathrm{i}dy + \frac{(1 + v_x^2)dx + v_x v_y dy}{W} + \mathrm{i}\,\frac{v_x v_y dx + (1 + v_y^2)dy}{W}$$

where $W = \sqrt{1 + v_x^2 + v_y^2}$. Alternately, one could take this as a definition of F. (The minimal surface equation implies that dF is closed.) With this definition, the proof of Proposition 2 is more computational but avoids Weierstrass representation.

3. CLASSICAL EXAMPLES

We focus on examples which are bounded by closed curves, as they are probably more interesting from the hydrodynamics point of view. All these examples admit deformations, and a lot more examples are known, see [9]. The Weierstrass data for all these examples is explicit and one can compute numerically the corresponding domain with hollow vortices: see Figure 2.

(1) The horizontal Scherk surface, a periodic minimal surface with horizontal period:

$$g = \frac{1}{z}, \quad \omega = \frac{z\,dz}{z^4 + 6z^2 + 1}.$$

(2) Karcher toroidal halfplane layers, a family of doubly periodic minimal surfaces with one horizontal and one vertical periods:

$$g = \frac{1}{z}, \quad \omega = \frac{dz}{\sqrt{(z^2 + a^2)(z^2 + a^{-2})}}, \quad 0 < a < 1.$$

(3) Schwarz P-surface, a triply periodic minimal surface:

$$g = \frac{1}{z}, \quad \omega = \frac{z\,dz}{\sqrt{z^8 - 14z^4 + 1}}.$$

(4) Schwarz H-surfaces, a family of triply periodic minimal surfaces:

$$g = \frac{1}{z}, \quad \omega = \frac{z\,dz}{\sqrt{z(z^3 + a^3)(z^3 + a^{-3})}}, \quad 0 < a < 1.$$

In all these examples, the Riemann surface Σ is a branched cover of the unit disk $D(0, 1)$ punctured at $z = \pm\mathrm{i}(\sqrt{2} - 1)$ in Case (1) and $z = 0$ in Case (2). The residues at the punctures and the multivaluation of the square roots are responsible for the periods of the minimal surfaces and the corresponding domains Ω.
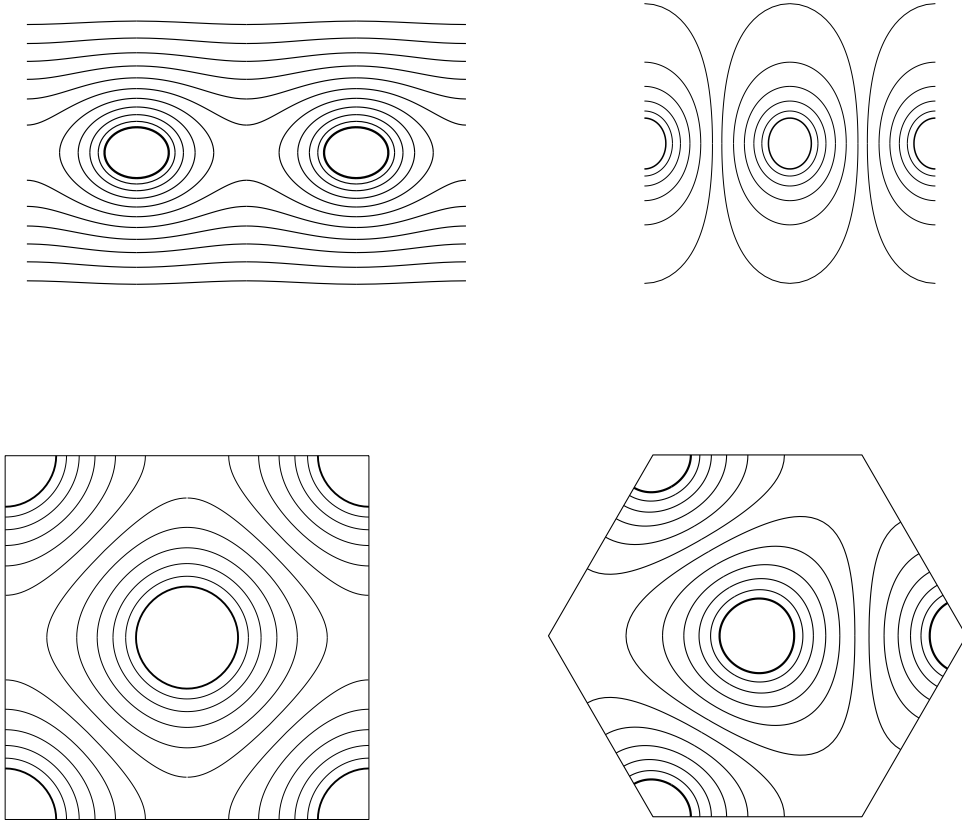
FIGURE 2. Top: the periodic domains corresponding to the horizontal Scherk surface (left) and a toroidal half-plane layer with $a = 0.5$ (right). Bottom: the doubly-periodic domains corresponding to Schwarz P-surface (left) and H-surface with $a = 0.5$ (right). The curves are the stream lines. Computed with Maple.

4. DOMAINS WITH SMALL HOLLOW VORTICES

In this section, we give a general method to construct domains with small hollow vortices, first in the finite connectivity case, then in the periodic case. In what follows, we identify points and vectors in the plane $\mathbb{R}^2$ with complex numbers.

4.1. **Domains with a finite number of hollow vortices.** First some definitions. Let (Ω, u) be a solution to Problem (1). Let γ be a closed curve in the boundary of Ω. Let $\mathcal{C}(\gamma) = \int_\gamma \vec{v} \cdot \vec{d\ell}$ be the circulation of $\vec{v}$ on the curve γ. Since $||\vec{v}|| = 1$ on γ, the absolute value of $\mathcal{C}(\gamma)$ is equal to the length of γ, but it can have either sign, depending on whether the vortex is "spinning" left or right.

Definition 1. *A vortex configuration is a finite set of $n \geq 2$ distinct points $p_1, \cdots, p_n$ in the complex plane with weights $c_1, \cdots, c_n$ which are non-zero real numbers. Forces are defined by*

$$F_i = \sum_{j \neq i} \frac{c_i c_j}{p_i - p_j}.$$

We say a configuration is balanced if $F_i = 0$ for $i = 1, \cdots, n$. We say a balanced configuration is non-degenerate if the $n \times n$ jacobian matrix $\frac{\partial F_i}{\partial p_j}$ has complex rank $n - 2$.

Observe that we always have

$$(4) \qquad \qquad \sum_{i=1}^{n} F_i = 0$$

$$(5) \qquad \qquad \sum_{i=1}^{n} p_i F_i = \sum_{i<j} c_i c_j.$$

Hence $n - 2$ is the maximum rank that the jacobian matrix may have. Also, (5) gives a restriction on the weights for a balanced configuration to exist.

Theorem 2. *Given a balanced, non-degenerate configuration, there exists a 1-parameter family of solutions (Ω_t, u_t) of the hollow vortex problem (1), depending on a small parameter $t > 0$, such that:*

(1) Ω_t has n boundary components, denoted $\gamma_{1,t} \cdots, \gamma_{n,t}$, all of them closed curves.
(2) The circulation $\mathcal{C}(\gamma_{i,t})$ is equal to $2\pi c_i t$.
(3) As $t \to 0$, $\gamma_{i,t}$ shrinks to the point p_i. Moreover, its asymptotic shape is circular.

Here is a simple example of balanced configuration, with dihedral symmetry of order $n - 1$ ($n \geq 3$):

$$c_j = 1, \quad p_j = e^{2\pi \mathrm{i}j/(n-1)} \quad \text{for } 1 \leq j \leq n - 1,$$
$$c_n = 1 - \frac{n}{2}, \quad p_n = 0.$$

The configuration is balanced by symmetry and Equation (5). One can easily break the symmetries by perturbing the weights.

The proof of Theorem 2 follows [13] in the minimal case and is omitted. It is also very similar to the proof of Theorem 3 below, which we give in Appendix C.

4.2. **Periodic domains.** In this section, we construct periodic domains with hollow vortices and period T in the x-direction, and such that the velocity vector has a limit as $y \to \pm\infty$. The limit velocities cannot be arbitrary, as the following proposition shows:

Proposition 3. *Let Ω be a domain with hollow vortices. Assume that*

(1) Ω and the velocity vector $\vec{v}$ are periodic with period T in the x-direction.
(2) The quotient Ω/T has a finite number of boundary components, denoted $\gamma_1, \cdots, \gamma_n$, all of them closed curves.

(3) *The velocity vector $\vec{v}$ has a limit as $y \to +\infty$ and $y \to -\infty$, denoted respectively v^+ and v^-.*

Then either:

(a) $v^+ = v^-$ *and* $\sum_{i=1}^{n} \mathcal{C}(\gamma_i) = 0$, *or*

(b) $v^+ = -v^- = \dfrac{1}{2T} \sum_{i=1}^{n} \mathcal{C}(\gamma_i)$.

Here we see the vectors v^+ and v^- as complex numbers. In Case (b), the limit velocity must be real numbers, while there is no such restriction in Case (a).

Proof. Since we identify vectors with complex numbers, we can write $\vec{v} = -2iu_{\bar{z}}$. Hence

$$\lim_{y \to \pm\infty} 2u_z = -i\overline{v^{\pm}}.$$

If γ is a stream line, the circulation of the velocity is related to u_z by

(6) $$\int_{\gamma} 2u_z \, dz = \int_{\gamma} u_x \, dx + u_y \, dy - i \int_{\gamma} u_y \, dx - u_x \, dy = -i\,\mathcal{C}(\gamma).$$

For large R, consider the domain $\Omega_R = \Omega \cap \{-R < y < R\}$. By Cauchy Theorem,

$$0 = \int_{\partial(\Omega_R/T)} 2u_z \, dz = \sum_{i=1}^{n} \int_{\gamma_i} 2u_z \, dz + \int_{z=-iR}^{T-iR} 2u_z \, dz + \int_{z=T+iR}^{iR} 2u_z \, dz.$$

We let $R \to \infty$ and obtain

(7) $$\sum_{i=1}^{n} \mathcal{C}(\gamma_i) = T(\overline{v^+} - \overline{v^-}).$$

Since u is constant on γ_i, we have $du = u_z \, dz + u_{\bar{z}} \, d\bar{z} = 0$ along γ_i. Hence using $|2u_z| = 1$ on γ_i,

$$\int_{\gamma_i} (2u_z)^2 dz = -\int_{\gamma_i} 4u_z \overline{u_z} \, d\bar{z} = -\int_{\gamma_i} d\bar{z} = 0.$$

Using Cauchy theorem again with the function $(2u_z)^2$ and letting $R \to \infty$ gives

(8) $$T((\overline{v^+})^2 - (\overline{v^-})^2) = 0.$$

Proposition 3 follows from (7) and (8). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 2. *A periodic vortex configuration is a finite set of non-zero complex numbers $p_1, \cdots, p_n$ with weight $c_1, \cdots, c_n$ which are non-zero real numbers, together with a non-zero complex number c_0. We assume that either:*

        Case (a) $c_1 + \cdots + c_n = 0$, or
        Case (b) $c_1 + \cdots + c_n + 2c_0 = 0$.

We define forces by

$$F_i = \sum_{j \neq i} c_i c_j \frac{p_i + p_j}{p_i - p_j} + \begin{cases} 2c_i c_0 & \text{in Case (a)} \\ 0 & \text{in Case (b)} \end{cases}$$

We say a configuration is balanced if $F_i = 0$ for $1 \leq i \leq n$. We say a balanced configuration is non-degenerate if the jacobian matrix $\frac{\partial F_i}{\partial p_j}$ has complex rank $n - 1$.

Observe that in either case, $F_1 + \cdots + F_n = 0$, so $n - 1$ is the maximum rank that the jacobian matrix may have.

Theorem 3. *Given a balanced, non-degenerate periodic configuration, there exists a 1-parameter family of solutions (Ω_t, u_t) of the hollow vortex problem (1), depending on a small parameter $t > 0$, such that:*

(1) *Ω_t is a periodic domain with period $T = 2\pi$.*
(2) *The quotient Ω_t/T has n boundary components, denoted $\gamma_{1,t} \cdots, \gamma_{n,t}$, all of them closed curves.*
(3) *The circulation $\mathcal{C}(\gamma_{i,t})$ is equal to $2\pi c_i t$.*
(4) *As $t \to 0$, $\gamma_{j,t}$ shrinks to the point $q_j = \mathrm{i} \log p_j$. Moreover, its asymptotic shape is circular.*
(5) *In Case (a), the limit of the velocity as $y \to \pm\infty$ is $t\overline{c_0}$. In Case (b), the limit of the velocity as $y \to \pm\infty$ is $\mp t c_0$.*

We prove this theorem in Appendix C. The proof follows [2] in the minimal case. Please take care that the limit position of the vortices is $q_j = \mathrm{i} \log p_j$ and not p_j as in Theorem 2. The $2\pi\mathrm{i}$ multivaluation of the complex logarithm is responsible for the period $T = 2\pi$ of the domain. In term of the points q_j, the forces are given by

$$F_i = -\mathrm{i} \sum_{j \neq i} c_i c_j \cot \frac{q_i - q_j}{2} + \begin{cases} 2c_i c_0 & \text{in Case (a)} \\ 0 & \text{in Case (b)} \end{cases}$$

4.3. **Examples of periodic configurations of type (a).** We take $n = 2$, $c_1 = 1$ and $c_2 = -1$. Solving $F_1 = 0$ gives

$$q_2 = q_1 + \mathrm{i} \log \frac{2c_0 - 1}{2c_0 + 1}, \qquad c_0 \neq \pm 0.5.$$

The points q_i for various values of c_0 are represented on Figures 3, 4, 5.

If we take $n = 5$, $c_1 = c_2 = c_3 = 1$ and $c_4 = c_5 = -1.5$, we obtain an uneven vortex street: see Figure 6.

4.4. **Examples of periodic configurations of type (b).** First assume that all c_i are equal to 1. The configuration $p_j = e^{2\pi \mathrm{i} j / n}$ is balanced by symmetry: all forces F_i are equal and their sum is zero. This gives $q_j = -2\pi j / n$ so the vortices are regularly spaced. This configuration gives the family of domains corresponding to the family of horizontal Scherk surfaces when it is close to its catenoidal limit.
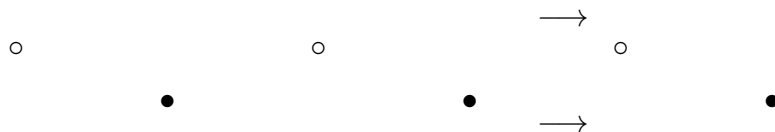
FIGURE 3. A periodic configuration of type (a) with $n = 2$, $c_0 = 0.25$. Circles represent vortices with right spin ($c_i > 0$), bullets represent vortices with left spin ($c_i < 0$). The arrows indicate the direction of the velocity at infinity. Three fundamental domains are represented. Compare with Figure 1.
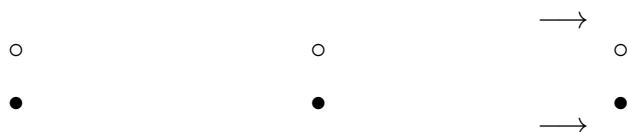
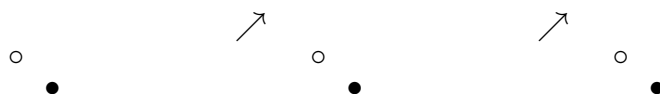FIGURE 4. A periodic configuration of type (a) with $n = 2$, $c_0 = 1$.

FIGURE 5. A periodic configuration of type (a) with $n = 2$, $c_0 = e^{-i\pi/4}$.
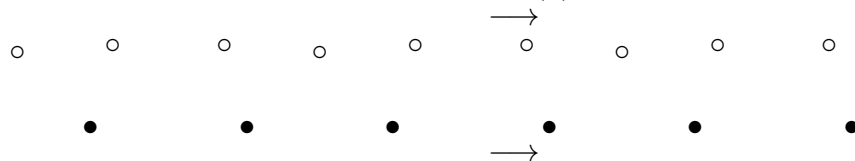
FIGURE 6. A periodic configuration of type (a) with $n = 5$, $c_1 = c_2 = c_3 = 1$, $c_4 = c_5 = -1.5$ and $c_0 = 0.5$. Computed with Maple.
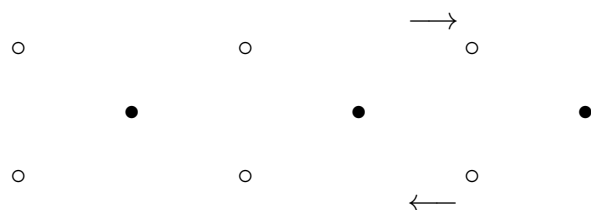
FIGURE 7. A periodic configuration of type (b) with $n = 3$, $c_1 = c_2 = 1$ and $c_3 = -1.5$.

To get a more interesting example, take $n = 3$, $c_1 = c_2 = 1$ and leave c_3 as a parameter. We may normalize $p_3 = 1$. Computations show that p_1 and p_2 are the roots of the polynomial $P(z) = z^2 + \dfrac{2c_3}{c_3 + 1}z + 1$. We obtain a two lanes vortex street, see Figure 7.

Appendix A. Proof of Proposition 1

Proof of Point (1):

$$x_3(z) = u(z_0) + \operatorname{Re} \int_{z_0}^{z} 2u_z \, dz = u(z_0) + \int_{z_0}^{z} (u_z \, dz + u_{\overline{z}} \, d\overline{z}) = u(z_0) + \int_{z_0}^{z} du = u(z).$$

Proof of Point (2): consider the differential

$$d\psi = dx_1 + \mathrm{i} dx_2 = \operatorname{Re}\left(\frac{1}{2}(g^{-1} - g)\omega\right) + \mathrm{i}\operatorname{Re}\left(\frac{\mathrm{i}}{2}(g^{-1} + g)\omega\right) = \frac{1}{2}(\overline{g^{-1}\omega} - g\omega).$$

With our choice of g and ω,

$$(9) \qquad d\psi = \frac{1}{2}\left(dz - 4(u_{\overline{z}})^2 \, d\overline{z}\right).$$

We have to prove that $d\psi$ is an exact differential (namely, the differential of a globally defined function ψ). If $t \mapsto \gamma(t)$ is a parametrization of a boundary component of Ω, then since u is constant on γ:

$$du(\gamma') = 0 = (u_z dz + u_{\overline{z}} d\overline{z})(\gamma').$$

$$(10) \qquad d\psi(\gamma') = \frac{1}{2}\left(dz(\gamma') + 4u_{\overline{z}}u_z \, dz(\gamma')\right) = \frac{1}{2}(1 + ||\nabla u||^2)dz(\gamma') = dz(\gamma').$$

Hence if γ is a closed component of $\partial\Omega$, $\int_{\gamma} d\psi = 0$. Since Ω is a planar domain, this implies that $d\psi$ is an exact differential. Also (10) proves Point (3).

Proof of Point (4): We prove that

$$(11) \qquad |2(\psi(z') - \psi(z)) - (z' - z)| < |z' - z|$$

which implies Point (4) by triangular inequality. We may decompose the segment $[z, z']$ into n segments $[z_i, z_{i+1}]$ for $1 \le i \le n$, such that $z_1 = z$, $z_{n+1} = z'$, $z_i \in \partial\Omega$ for $2 \le i \le n$ and for each i, the open segment (z_i, z_{i+1}) is either included in Ω or its complement. In the first case, we have by Equation (9) and using $||\nabla u|| < 1$

$$(12) \quad |2(\psi(z_{i+1}) - \psi(z_i)) - (z_{i+1} - z_i)| = \left|\int_{z_i}^{z_{i+1}} 4(u_{\overline{z}})^2 \, d\overline{z}\right| \le \int_{z_i}^{z_{i+1}} 4|u_{\overline{z}}|^2 \, |dz| < |z_{i+1} - z_i|.$$

In the second case, since Ω is a concave domain, z_i and z_{i+1} must be on the same component of $\partial\Omega$. By Point (3), $\psi(z_{i+1}) - \psi(z_i) = z_{i+1} - z_i$ so (12) becomes an equality. Summing for $1 \le i \le n$ gives (11).

Proof of Point (5): Equation (9) and $||\nabla u|| < 1$ implies that $d\psi$ is an isomorphism so ψ is a local diffeomorphism. Point (4) implies that ψ is injective, so is a global diffeomorphism onto its image.

Proof of Point (6): since ψ is a homeomorphism from Ω to $\widehat{\Omega}$ and extends continuously to $\overline{\Omega}$, we have $\psi(\partial\Omega) \subset \partial\widehat{\Omega}$ by elementary topology. (Here $\overline{\Omega}$ denotes the closure of Ω.) Assume by contradiction that $\psi(\partial\Omega) \neq \partial\widehat{\Omega}$ and let $a_0 \in \partial\widehat{\Omega} \setminus \psi(\partial\Omega)$.

The finiteness hypothesis (Hypothesis 1) ensures that $\psi(\partial\Omega)$ is closed. Indeed, for each component γ of $\partial\Omega$, $\psi(\gamma)$ is a translate of γ so is closed, and the finiteness hypothesis prevents them from accumulating.

Let $\varepsilon = d(a_0, \psi(\partial\Omega))$. Choose a point $a_1 \in \widehat{\Omega}$ such that $|a_0 - a_1| \leq \frac{\varepsilon}{4}$. Let a_2 be a point on $\partial\widehat{\Omega}$ whose distance to a_1 is minimum. Then $d(a_2, \psi(\partial\Omega)) \geq \frac{\varepsilon}{2}$ and the semi-open segment $[a_1, a_2)$ is entirely included in $\widehat{\Omega}$. Let $\alpha(t) : [0, \ell) \to \Omega$ be a path such that $\psi(\alpha(t))$ is the parametrization at unit speed of the segment $[a_1, a_2]$. We must have $||\alpha(t)|| \to \infty$ as $t \to \infty$, else a_2 would be in $\psi(\overline{\Omega})$. This implies that the path α has infinite length. Now the conformal metric induced by the minimal immersion X is given by

$$ds = \frac{1}{2}(|g| + |g|^{-1})|\omega| \geq \frac{1}{2}|dz|.$$

Hence, the curve $X(\alpha(t))$ on M has infinite length. This curve is the graph of the function v on the segment $[a_1, a_2]$. By standard results in minimal surface theory (see the proof of Lemma 2 in [14] for the details), this implies that $\lim_{z \to a_2} v(z) = \pm\infty$. Moreover, there exists a divergence line L, containing a_2 and contained in $\partial\widehat{\Omega}$, such that $v \to \pm\infty$ on L. By connectedness, $\widehat{\Omega}$ must be on one side of L. To derive a contradiction, we distinguish two cases:

- If $d(L, \psi(\partial\Omega)) > 0$, then the function v satisfies the minimal surface equation in a band with boundary value $\pm\infty$ on one side. This is impossible by Proposition 1 in [12].

  Remark 3. *If all components of $\partial\Omega$ are closed curves, then the finiteness hypothesis implies that $d(L, \psi(\partial\Omega)) > 0$.*

- If $d(L, \psi(\partial\Omega)) = 0$, then there is an unbounded component of $\partial\Omega$, say γ_1, such that $\psi(\gamma_1)$ is asymptotic to L. Also, there can be at most two such components. Label γ_2 the other component asymptotic to L, if any. Let us write $\widehat{\gamma}_i = \psi(\gamma_i)$. There exists $\varepsilon > 0$ so that all other components of $\psi(\partial\Omega)$ are at distance greater than ε of L. We obtain a contradiction using the catenoid as a barrier as in the proof of the strong half-space theorem of Hoffman Meeks [7]. The only difference is that M is not complete so we have to mind its boundary.

  Without loss of generality, we may assume that L is the line $x_2 = 0$ in the horizontal plane, M lies in the half-space $x_2 > 0$, and also $v < 0$ on $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ and $v \to +\infty$ on L. Let ν be the interior conormal to the boundary of M. Since $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ are horizontal symmetry curves, ν is vertical. Evaluating the vertical flux in the subdomain of $\widehat{\Omega}$ defined by $R < |x_1| < 2R$ and $0 < x_2 < \varepsilon$ for large values of R, we obtain that $\nu = (0, 0, 1)$ on $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$.

  Let C be the horizontal half-catenoid $x_1^2 + x_3^2 = \cosh^2 x_2$, $x_2 < 0$. Let $C_t = (0, a, 0) + t\, C_1$, where $0 < a < \varepsilon$ and $0 < t \leq 1$. If a is small enough then C_1 does not intersect M. Also, as $t \to 0$, C_t converges to the vertical plane $x_2 = a$, so C_t

intersects M for $t > 0$ small enough. Let $t_0 < 1$ be the largest time so that C_t intersects M. Then C_{t_0} intersects M at a boundary point. Since C_t lies in the half-space $x_2 < a < \varepsilon$, that point must be on $\widehat{\gamma}_1$ or $\widehat{\gamma}_2$. Since $x_3 < 0$ on $\widehat{\gamma}_i$ and $\nu = (0, 0, 1)$, M will still intersect C_t for t slightly larger than t_0, a contradiction. $\square$

APPENDIX B. PROOF OF PROPOSITION 2

Proof of Point (1): consider the differential $d\varphi = -g\omega$ on Σ. We have to prove that $d\varphi$ is an exact differential. Since ω has a zero at each pole of g, $d\varphi$ is holomorphic in Σ. Let γ be a component of $\partial\Omega$. Since $|g| = 1$ on γ and $\omega(\gamma')$ is imaginary,

$$(13) \qquad d\psi(\gamma') = \frac{1}{2}\left(\overline{g^{-1}\omega(\gamma')} - g\omega(\gamma')\right) = -g\omega(\gamma') = d\varphi(\gamma').$$

Since $d\psi$ is an exact differential and Σ is diffeomorphic to a planar domain, $d\varphi$ is the differential of a globally defined function φ. Then $F = \varphi \circ \psi^{-1}$ is well defined. Point (2) is a consequence of (13).

Proof of Point (3): let τ be the unit vector in the direction of $z' - z$. We prove that

$$(14) \qquad \langle F(z') - F(z), \tau \rangle > \langle z' - z, \tau \rangle$$

which implies Point (3). We may decompose the segment $[z, z']$ into n segments $[z_i, z_{i+1}]$ for $1 \le i \le n$, such that $z_1 = z$, $z_{n+1} = z'$, $z_i \in \partial\widehat{\Omega}$ for $2 \le i \le n$ and for each i, the open segment (z_i, z_{i+1}) is either included in $\widehat{\Omega}$ or its complement. In the first case, let $\alpha(t) : (0, \ell) \to \Sigma$ be such that $\psi(\alpha(t))$ is the parametrization of the segment (z_i, z_{i+1}) at constant speed τ, in other words $d\psi(\alpha') = \tau$. Then

$$\begin{aligned} \langle d\varphi(\alpha'), \tau \rangle &= \langle d\varphi(\alpha'), d\psi(\alpha') \rangle \\ &= \langle d\varphi(\alpha') - d\psi(\alpha'), d\psi(\alpha') \rangle + ||d\psi(\alpha')||^2 \\ &= \frac{1}{4}\left(|g\omega(\alpha')|^2 - |g^{-1}\omega(\alpha')|^2\right) + 1 \\ &> 1 \quad \text{since } |g| > 1. \end{aligned}$$

Integrating from $t = 0$ to ℓ, we obtain

$$(15) \qquad \langle F(z_{i+1}) - F(z_i), \tau \rangle > \ell = \langle z_{i+1} - z_i, \tau \rangle.$$

If the segment (z_i, z_{i+1}) is included in the complementary of $\widehat{\Omega}$, then since $\widehat{\Omega}$ is a concave domain, z_i and z_{i+1} must be on the same component of $\partial\widehat{\Omega}$, so $F(z_i) = F(z_{i+1})$ by Point (2). Hence (15) become an equality. Summing for $1 \le i \le n$ gives (14).

Proof of Point (4): since $|g| > 1$ in Σ, $d\varphi \neq 0$ in Σ so φ and F are local diffeomorphisms. By Point (3), F is injective, so is a global diffeomorphism onto its image Ω. By Point (3), F is proper, so $\partial\Omega = F(\partial\widehat{\Omega})$.

Proof of Point (5): we have

$$u = v \circ F^{-1} = v \circ \psi \circ \varphi^{-1} = x_3 \circ \varphi^{-1}.$$

Since x_3 is harmonic and φ is biholomorphic, u is a harmonic function. Differentiating $u(\varphi(z)) = x_3(z)$, we obtain

$$2u_z(\varphi(z)) \times (-g(z)\omega) = 2\frac{\partial x_3}{\partial z}dz = \omega.$$

Hence

$$2u_z(\varphi(z)) = \frac{-1}{g(z)}$$

which implies that $||\nabla u|| < 1$ in Ω and $||\nabla u|| = 1$ on $\partial\Omega$. $\square$

Appendix C. Proof of Theorem 3

We need to construct a meromorphic function g and a holomorphic differential ω on a domain $\Sigma \subset \mathbb{C}$ such that $|g| > 1$ in Σ, $|g| = 1$ on $\partial\Sigma$ and ω is imaginary along $\partial\Sigma$. Then we proceed as in Section 2.4 for the correspondence "minimal $\to$ vortex". Everything depends on the small parameter $t > 0$.

C.1. **The domain Σ_t and the function g_t.** Consider the function

$$f(z) = c_0 + \sum_{i=1}^{n} \frac{a_i z}{z - p_i}.$$

Here $a_1, \cdots, a_n$ are non-zero complex parameters such that

$$(16) \qquad\qquad a_1 + \cdots + a_n = \begin{cases} 0 & \text{in Case (a)} \\ -2c_0 & \text{in Case (b)} \end{cases}$$

For $t > 0$, let Σ_t be the domain $|tf(z)| < 1$. For t small enough, Σ_t has n boundary components which we label $\gamma_1, \cdots, \gamma_n$. We define the meromorphic function g_t on Σ_t by

$$g_t(z) = \frac{-\mathrm{i}}{tf(z)}.$$

We have $|g_t| > 1$ in Σ_t and $|g_t| = 1$ on $\partial\Sigma_t$.

C.2. **Opening nodes.** To define the holomorphic differential ω_t we need to construct the "double" of the domain Σ_t. We do this by "opening nodes". Consider two copies of the complex plane, denoted $\mathbb{C}_1$ and $\mathbb{C}_2$. As f has a simple pole at p_i, there exists a neighborhood $V_i \subset \mathbb{C}_1$ of p_i, a neighborhood $W_i \subset \mathbb{C}_2$ of $\overline{p_i}$ and $\varepsilon > 0$ such that the holomorphic functions

$$v_i(z) = \frac{1}{f(z)} : V_i \to D(0, \varepsilon) \quad \text{and} \quad w_i(z) = \frac{1}{\overline{f(\overline{z})}} : W_i \to D(0, \varepsilon)$$

are biholomorphic. Consider the disjoint union $\mathbb{C}_1 \cup \mathbb{C}_2$. For $i = 1, \cdots, n$, remove the disks $|v_i| < \frac{t^2}{\varepsilon}$ and $|w_i| < \frac{t^2}{\varepsilon}$. Identify the point $z \in V_i$ with the point $z' \in W_i$ such that

$$(17) \qquad\qquad v_i(z)w_i(z') = t^2.$$

This defines a Riemann surface of genus $n - 1$ which we denote $\widehat{\Sigma}_t$. We also define Σ_0 as the (singular) Riemann surface with n nodes (or double points) obtained by identifying $p_i \in \mathbb{C}_1$ with $\overline{p_i} \in \mathbb{C}_2$ for $1 \leq i \leq n$.

The anti-holomorphic involution σ which exchanges $z \in \mathbb{C}_1$ with $\overline{z} \in \mathbb{C}_2$ is well defined on $\widehat{\Sigma}_t$ by the following computation:

$$z \sim z' \Rightarrow v_i(z)w_i(z') = t^2 \Rightarrow \overline{v_i(z)w_i(z')} = t^2 \Rightarrow w_i(\overline{z})v_i(\overline{z'}) = t^2 \Rightarrow \sigma(z) \sim \sigma(z').$$

We see Σ_t as a domain in $\mathbb{C}_1$. The involution σ exchanges the disjoint domains $\Sigma_t \subset \mathbb{C}_1$ and $\overline{\Sigma_t} \subset \mathbb{C}_2$ and its fixed set is $\partial \Sigma_t$, indeed:

$$z = \sigma(z) \Leftrightarrow z \sim \overline{z} \Leftrightarrow v_i(z)w_i(\overline{z}) = t^2 \Leftrightarrow |tf(z)|^2 = 1.$$

Hence we can see $\widehat{\Sigma}_t$ as the disjoint union of Σ_t and its mirror image $\overline{\Sigma_t}$ glued along their boundaries by the map $z \mapsto \overline{z}$. In other words, $\widehat{\Sigma}_t$ is the double of Σ_t. The point of constructing $\widehat{\Sigma}_t$ by opening nodes is that it will allow us to understand the limit $t \to 0$. Finally, we can extend the definition of the function g_t to $\widehat{\Sigma}_t$ by

$$g_t(z) = \begin{cases} \dfrac{-\mathrm{i}}{tf(z)} = -\mathrm{i}\,\dfrac{v_i(z)}{t} & \text{in } \mathbb{C}_1 \\[2mm] -\mathrm{i}\,t\overline{f(\overline{z})} = -\mathrm{i}\,\dfrac{t}{w_i(z)} & \text{in } \mathbb{C}_2 \end{cases}$$

The identification (17) implies that g_t is well defined on $\widehat{\Sigma}_t$. Moreover, g_t has the symmetry $g_t \circ \sigma = 1/\overline{g}$.

C.3. **The holomorphic differential** ω_t. We compactify $\widehat{\Sigma}_t$ by adding the points at infinity in $\mathbb{C}_1$ and $\mathbb{C}_2$, denoted ∞_1 and ∞_2. We also denote 0_1 and 0_2 the points $z = 0$ in $\mathbb{C}_1$ and $\mathbb{C}_2$.

Proposition 4. *For $t \neq 0$, there exists a unique meromorphic differential ω_t on $\widehat{\Sigma}_t$ with 4 simple poles at 0_1, 0_2, ∞_1 and ∞_2, such that*

$$\int_{\gamma_j} \omega_t = -2\pi \mathrm{i} c_j \quad \text{for } 1 \leq j \leq n$$
$$\mathrm{Res}_{0_1} \omega_t = c_0$$
$$\mathrm{Res}_{0_2} \omega_t = -\overline{c_0}.$$

Here $c_0, c_1, \cdots, c_n$ are given from the configuration. Moreover:

(1) ω_t has the following symmetry: $\sigma^* \omega_t = -\overline{\omega_t}$.

(2) ω_t *extends analytically at* $t = 0$ *with*

$$(18) \qquad \omega_0 = \frac{c_0 \, dz}{z} + \sum_{i=1}^{n} \frac{c_i \, dz}{z - p_i} \qquad in \ \mathbb{C}_1$$

(3) *The residues of* $g_t \omega_t$ *at* 0_1 *and* ∞_1 *are given by*

$$(19) \qquad \mathrm{Res}_{0_1} g_t \omega_t = \frac{-\mathrm{i}}{t}$$

$$(20) \qquad \mathrm{Res}_{\infty_1} g_t \omega_t = \frac{\mathrm{i}}{t}$$

Proof: the existence of ω_t follows from the standard theory of compact Riemann surface: the curves $\gamma_1, \cdots, \gamma_{n-1}$ are the A-cycles of a canonical homology basis. In general, one can define a meromorphic differential with simple poles by prescribing its periods on these cycles and the residues at the poles, with the only restriction that the sum of the residues is zero. In our case, prescribing the residue at ∞_1 is the same as prescribing the period along the last cycle γ_n.

Proof of Point (1):

$$\int_{\gamma_j} \overline{\sigma^* \omega_t} = \int_{\sigma(\gamma_j)} \overline{\omega_t} = -\int_{\gamma_j} \overline{\omega_t} = 2\pi \mathrm{i} c_j,$$

$$\mathrm{Res}_{0_2} \overline{\sigma^* \omega_t} = \overline{\mathrm{Res}_{0_1} \omega_t} = \overline{c_0}, \qquad \mathrm{Res}_{0_1} \overline{\sigma^* \omega_t} = -c_0.$$

Hence the meromorphic differentials $\overline{\sigma^* \omega_t}$ and $-\omega_t$ have the same poles, periods and residues, so they are equal.

Proof of Point (2): we know from the theory of opening nodes that ω_t extends analytically at $t = 0$, and ω_0 is a meromorphic differential on Σ_0 with at most simples poles at the nodes (see [11], [13] or [15]). The residues at the poles are determined by the prescribed periods. (Here γ_i is oriented as a boundary of Σ_t so has clockwise orientation.)

Proof of Point (3): we have

$$f(0) = c_0, \quad g_t(0_1) = \frac{-\mathrm{i}}{t c_0}, \quad \mathrm{Res}_0 \omega_t = c_0.$$

This gives (19). In Case (a), we have, using $a_1 + \cdots + a_n = c_1 + \cdots + c_n = 0$,

$$f(\infty) = c_0, \quad g_t(\infty_1) = \frac{-\mathrm{i}}{t c_0}, \quad \mathrm{Res}_{\infty_1} \omega_t = -c_0.$$

In Case (b), we have, using $a_1 + \cdots + a_n = c_1 + \cdots + c_n = -2c_0$,

$$f(\infty) = -c_0, \quad g_t(\infty_1) = \frac{\mathrm{i}}{t c_0}, \quad \mathrm{Res}_{\infty_1} \omega_t = c_0.$$

In either cases, this gives (20). $\square$

C.4. **Zeros of ω_t.**

Proposition 5. *For t small enough, one can adjust the parameters $a_1, \cdots, a_n$ so that ω_t has a zero at each zero and pole of g_t. Moreover, $a_i(t)$ is a smooth function of t and $a_i(0) = c_i$.*

Proof: the meromorphic differential ω_t has 4 poles on a genus $n-1$ compact Riemann surface so has $2n$ zeros. By symmetry, ω_t has n zeros in $\mathbb{C}_1$, which we call $\zeta_1(t), \cdots, \zeta_n(t)$. We have to solve $f(\zeta_i(t)) = 0$ for $1 \le i \le n$. We use the implicit function theorem at $t = 0$. When $t = 0$, an obvious solution is to take $a_i = c_i$ which gives by (18)

$$(21) \qquad\qquad \omega_0 = f(z) \frac{dz}{z}.$$

Then we compute

$$\frac{\partial f(\zeta_i)}{\partial a_j}\Big|_{t=0} = \frac{\zeta_i}{\zeta_i - p_j}.$$

The determinant of this $n \times n$ matrix is a Cauchy determinant so it is invertible. The problem to apply the implicit function theorem is that the parameters $a_1, \cdots, a_n$ are constrained by Equation (16), so we have in fact only $n-1$ parameters available. Let $h(a_1, \cdots, a_n, t) = (f(\zeta_1), \cdots, f(\zeta_{n-1}))$.

Lemma 1. *The partial differential of h with respect to $(a_1, \cdots, a_n)$ at $(c_1, \cdots, c_n, 0)$, restricted to the space $a_1 + \cdots + a_n = 0$, is an isomorphism.*

Proof: let $(a_1, \cdots, a_n)$ be in the kernel of the partial differential of h. Then

$$\sum_{j=1}^{n} \frac{a_i}{\zeta_i - p_j} = 0 \quad \text{for } 1 \le i \le n-1.$$

Consider the meromorphic differential on $\mathbb{C} \cup \{\infty\}$

$$\mu = \sum_{j=1}^{n} \frac{a_i}{z - p_j} \, dz.$$

Then μ has $n-1$ zeros at $\zeta_1, \cdots, \zeta_{n-1}$, n poles at $p_1, \cdots, p_n$ and is holomorphic at ∞ because $a_1 + \cdots + a_n = 0$. Hence $\mu = 0$ so $a_1 = \cdots = a_n = 0$. $\qquad\square$

By Lemma 1 and the implicit function theorem, we can solve $f(\zeta_i) = 0$ for $1 \le i \le n-1$. It remains to understand why $f(\zeta_n) = 0$. Let ζ' be the last zero of f in $\mathbb{C}_1$. The meromorphic differential $g_t \omega_t$ has 4 simple poles at $0_1, \infty_1, 0_2, \infty_2$, and at most a simple pole at ζ'. By Point (3) of Proposition 4 and by symmetry, the sum of the residues of $g_t \omega_t$ at $0_1, \infty_1, 0_2$ and ∞_2 is zero. By the residue theorem, the residue at ζ' is zero, so $g_t \omega_t$ is actually holomorphic at ζ', which means that $\zeta' = \zeta_n$. This proves Proposition 5. $\qquad\square$

Remark 4. *We tacitly assumed that the zeros $\zeta_1, \cdots, \zeta_n$ are distinct, which is the generic case. In case there are multiple zeros, the proof must be fixed using the Weierstrass preparation theorem. See details in [13].*

C.5. **The period problem.** From now on, we assume that $a_i(t)$ has the value given by Proposition 5.

Proposition 6. *For t small enough, one can adjust $p_1, \cdots, p_n$ so that for $1 \le i \le n$,*

$$(22) \qquad \int_{\gamma_i} g_t \omega_t = 0.$$

Moreover, $p_i(t)$ is a smooth function of t and $p_i(0)$ is given by the configuration.

Proof: by Point (3) of Proposition 4 and the residue theorem , we have

$$(23) \qquad \sum_{i=1}^n \int_{\gamma_i} g_t \omega_t = 2\pi i \left(\mathrm{Res}_{0_1} g_t \omega_t + \mathrm{Res}_{\infty_1} g_t \omega_t \right) = 0.$$

So it suffices to solve (22) for $1 \le i \le n-1$. By symmetry and definition of g_t,

$$\int_{\gamma_i} g_t \omega_t = \int_{\gamma_i} \frac{1}{\overline{g_t}} (-\overline{\omega_t}) = \mathrm{i} t \int_{\gamma_i} \overline{f \omega_t}.$$

So we want to solve

$$(24) \qquad \int_{\gamma_i} f \omega_t = 0 \quad \text{for } 1 \le i \le n-1.$$

We solve (24) using the implicit function theorem at $t = 0$. We compute

$$
\begin{aligned}
\int_{\gamma_i} f \omega_0 &= \int_{\gamma_i} f^2(z) \frac{dz}{z} \qquad \text{using (21)} \\
&= 2\pi i \, \mathrm{Res}_{p_i} \left(c_0 + \sum_{j=1}^n \frac{c_j z}{z - p_j} \right)^2 \frac{dz}{z} \\
&= 2\pi i \, \mathrm{Res}_{p_i} \left[\frac{c_i^2 z}{(z - p_i)^2} + \frac{2c_i}{z - p_i} \left(c_0 + \sum_{j \ne i} \frac{c_j z}{z - p_j} \right) \right] \\
&= 2\pi i \left(c_i^2 + 2c_0 c_i + 2 \sum_{j \ne i} \frac{c_i c_j p_i}{p_i - p_j} \right) \\
&= 2\pi i \left[c_i^2 + 2c_0 c_i + \sum_{j \ne i} c_i c_j \left(\frac{p_i + p_j}{p_i - p_j} + 1 \right) \right] \\
&= 2\pi i \left(\sum_{j=1}^n c_i c_j + 2c_0 c_i + \sum_{j \ne i} c_i c_j \frac{p_i + p_j}{p_i - p_j} \right) \\
&= 2\pi i F_i
\end{aligned}
$$

where F_i is as in Definition 2. Since the configuration is balanced and non-degenerate, we can solve (24) using the implicit function theorem. $\qquad \square$

Remark 5. *Since $F_1 + \cdots + F_n = 0$, it was crucial to have the relation (23) amongst the periods for all t.*

C.6. **Solution of the hollow vortex problem.** We proceed as in Appendix B, with ω replaced by $t\,\omega_t$. We define φ_t on $\Sigma_t \subset \mathbb{C}_1$ for $t \neq 0$ by

$$\varphi_t(z) = \mathrm{i} \log z_0 - \int_{z_0}^{z} g_t \, t \, \omega_t.$$

Proposition 7.   *(1) $\varphi_t : \Sigma_t \to \mathbb{C}/2\pi\mathbb{Z}$ is a well defined holomorphic map.*
    (2) φ_t extends analytically at $t = 0$ (away from the points $p_1, \cdots, p_n$) with $\varphi_0(z) = \mathrm{i} \log z$.
    (3) φ_t is a diffeomorphism onto its image $\Omega_t = \varphi_t(\Sigma_t) \subset \mathbb{C}/2\pi\mathbb{Z}$. (The domain Ω_t lifts to a periodic domain in the plane with period 2π.)
    (4) The function u_t defined on $\varphi_t(\Sigma_t)$ by

$$u_t(\varphi_t(z)) = \mathrm{Re} \int_{z_0}^{z} t \, \omega_t$$

    solves Problem (1) on Ω_t. Moreover, the velocity $\vec{v}_t$ and its circulation are given by

(25)
$$\vec{v}_t(\varphi_t(z)) = t\overline{f(z)}$$

$$\mathcal{C}(\varphi_t(\gamma_i)) = 2\pi \, t \, c_i.$$

Proof: by Propositions 5, $g_t\omega_t$ is holomorphic and non-zero in Σ_t. By Proposition 6, the only periods of $g_t\omega_t$ come from the residues at 0 and ∞. By Point (3) of Proposition 4, φ_t is well defined modulo 2π, which proves Point (1). To prove Point (2), we write

$$\varphi_t(z) = \mathrm{i} \log z_0 + \mathrm{i} \int_{z_0}^{z} \frac{\omega_t}{f}$$

and we use Equation (21). Regarding Point (3), we already know that φ_t is a local diffeomorphism because its derivative does not vanish. Consider a component γ_i of $\partial\Sigma_t$. The unit normal to $\varphi_t(\gamma_i)$ is g_t. Since the function g_t is a diffeomorphism from γ_i to the unit circle, $\varphi_t(\gamma_i)$ is a small convex curve. Consider then the well-defined holomorphic function $\psi = \exp(\mathrm{i}\varphi_t) : \Sigma_t \to \mathbb{C}$. Observe that ψ has a simple pole at 0 and a simple zero at ∞. Since each $\psi(\gamma_i)$ bounds a disk in the Riemann sphere, we can extend ψ into a local homeomorphism $\widetilde{\psi}$ from the Riemann sphere to itself. Since the Riemann sphere is compact and simply connected, $\widetilde{\psi}$ must be a diffeomorphism. Hence φ_t is injective, which proves Point (3).

Proof of Point (4): As in the proof of Point (5) of Proposition 2, we have

$$2\frac{\partial u_t}{\partial z}(\varphi_t(z)) = \frac{-1}{g_t(z)} = -\mathrm{i}\,t f(z).$$

Formula (25) for the velocity follows. From the definition of u_t we obtain

$$\varphi_t^*(2u_z\,dz) = t\omega_t.$$

By (6), the residue theorem and the definition of ω_t, we have

$$\mathcal{C}(\varphi_t(\gamma_i)) = \mathrm{i}\int_{\varphi_t(\gamma_i)} 2u_z\,dz = \mathrm{i}\int_{\gamma_i}\varphi_t^*(2u_z\,dz) = \mathrm{i}\int_{\gamma_i} t\omega_t = 2\pi\,t\,c_i.$$

$\square$

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